EECS 16A Designing Information Devices and Systems I
Spring 2023 Homework 14

This homework is due Friday, April 28th, 2023, at 23:59.
Self-grades are due Friday, May 5th, 2023, at 23:59.

Submission Format

Your homework submission should consist of **one** file.

- `hw14.pdf`: A single PDF file that contains all of your answers (any handwritten answers should be scanned).

Submit the file to the appropriate assignment on Gradescope.

1. Course Evaluation

Please fill out the course evaluation for EECS 16A by logging into `https://course-evaluations.berkeley.edu`. If at least 70% of the class fills out the course evaluation, everyone will receive one point of extra credit on the final, and if at least 80% fills it out, then everyone will receive two extra points.

2. Reading Assignment

For this homework, please read Notes 21 and 22. Note 21 introduces the concept behind GPS and returns to linear algebra with definitions of vector inner products, norms, orthogonality, and projections. Note 22 brings in the concept of correlation and its use for trilateration.

3. Mechanical Projections

Learning Goal: *The objective of this problem is to practice calculating projection of a vector and the corresponding squared error.*

(a) Find the projection of $\vec{b} = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$ onto $\vec{a} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$. What is the squared error between the projection and $\vec{b}$, i.e. $\|e\|^2 = \|\text{proj}_{\vec{a}}(\vec{b}) - \vec{b}\|^2$?

Solution:

$$\text{proj}_{\vec{a}}(\vec{b}) = \frac{\langle \vec{a}, \vec{b} \rangle}{\|\vec{a}\|^2}\vec{a} = \frac{\vec{a}^T\vec{b}}{\|\vec{a}\|^2}\vec{a} \tag{1}$$

First, compute $\|\vec{a}\|^2 = \langle \vec{a}, \vec{a} \rangle = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = 2$.

Second, compute $\langle \vec{a}, \vec{b} \rangle = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} = 2$.

Plugging in, $\text{proj}_{\vec{a}}(\vec{b}) = \frac{2\vec{a}}{2} = \vec{a}$.

The squared error between $\vec{b}$ and its projection onto $\vec{a}$ is $\|e\|^2 = \|\vec{a} - \vec{b}\|^2$

$\|e\|^2 = \| \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} \|^2$

$\|e\|^2 = \| \begin{bmatrix} -2 \\ -2 \\ 2 \end{bmatrix} \|^2$

$\|e\|^2 = 4 + 4 + 4 = 12.$

(b) Find the projection of $\vec{b} = \begin{bmatrix} 1 \\ 4 \\ -5 \end{bmatrix}$ onto the column space of $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$. What is the squared error between the projection and $\vec{b}$, i.e. $\|e\|^2 = \|\text{proj}_{\text{Col}(\mathbf{A})}(\vec{b}) - \vec{b}\|^2$?

Solution: Let $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $\vec{x} \in \mathbb{R}^2$ such that the projection of $\vec{b}$ onto the column space of $\mathbf{A}$ is $\mathbf{A}\vec{x}$.

We will compute $\hat{\vec{x}}$ by solving the following least squares problem,

$$\min_{\vec{x}} \|\mathbf{A}\vec{x} - \vec{b}\|^2 \tag{2}$$

The solution yields,

$$\hat{\vec{x}} = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\vec{b} \tag{3}$$

$$= \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}^T \begin{bmatrix} 1 \\ 4 \\ -5 \end{bmatrix} \tag{4}$$

$$= \begin{bmatrix} 0.5 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -4 \\ 4 \end{bmatrix} \tag{5}$$

$$= \begin{bmatrix} -2 \\ 4 \end{bmatrix} \tag{6}$$

Plugging in, the projection of $\vec{b}$ onto the column space of $\mathbf{A}$ is $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -2 \\ 4 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \\ -2 \end{bmatrix}$.

The squared error between the projection and $\vec{b}$ is $\|\vec{e}\|^2 = \left\| \begin{bmatrix} -2 \\ 4 \\ -2 \end{bmatrix} - \begin{bmatrix} 1 \\ 4 \\ -5 \end{bmatrix} \right\|^2 = 18.$

4. Mechanical Trilateration

Learning Goal: *The objective of this problem is to practice using trilateration to find the position based on distance measurements and known beacon locations.*

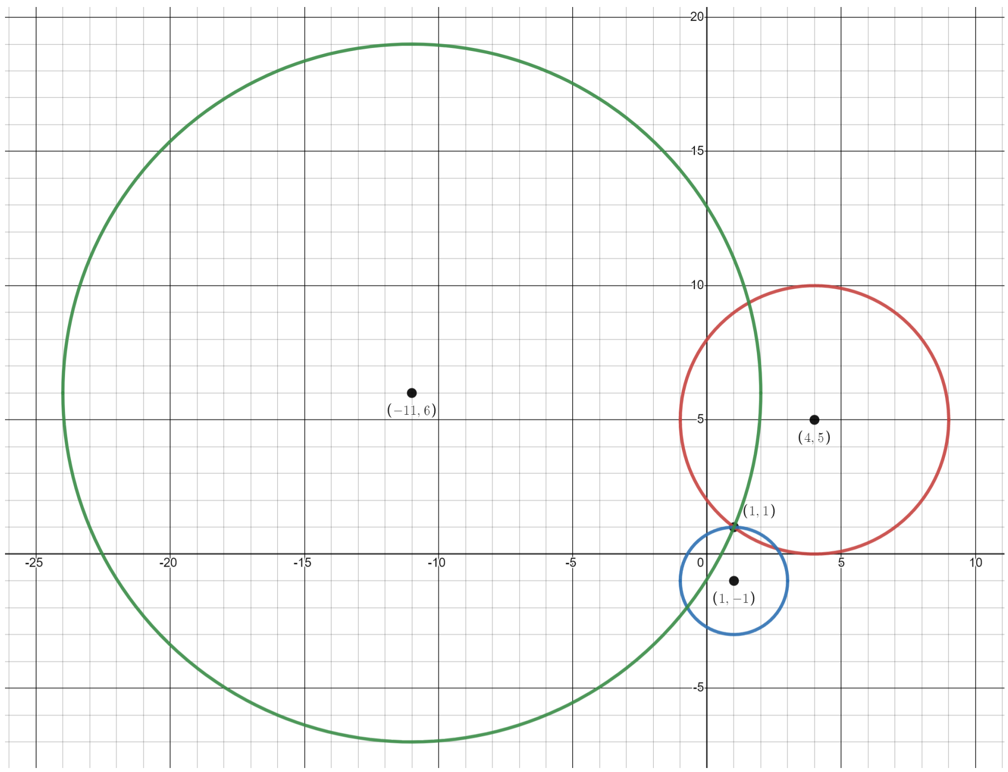
Trilateration is the problem of finding one's coordinates given distances from known beacon locations. For each of the following trilateration problems, you are given 3 beacon locations ($\vec{s}_1$, $\vec{s}_2$, $\vec{s}_3$) and the corresponding distance (d_1, d_2, d_3) from each beacon to your location.

For each problem, **graph** (by hand, with a graphing calculator, or iPython) the set of coordinates indicating your possible location for each beacon and find any coordinate solutions where they all intersect. Then **solve the trilateration problem algebraically** using the method introduced in lecture, to find your location or possible locations. If a solution does not exist, state that it does not.

For any solutions found using trilateration, be sure to check that they are consistent with the beacon measurements.

(a) $\vec{s}_1 = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$, $d_1 = 5$; $\vec{s}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $d_2 = 2$; $\vec{s}_3 = \begin{bmatrix} -11 \\ 6 \end{bmatrix}$, $d_3 = 13$

Solution: From graphing these equations we can see there is a single point of intersection, or in other words, a single possible solution to our location.



Now, we show a general approach to the trilateration problem, so that we can immediately write the linear system of equations for all three parts and solve for our solution algebraically. However, if you solved directly using concrete values, give yourself full credit.

$$\|\vec{x} - \vec{s}_1\|^2 = d_1^2$$
$$\|\vec{x} - \vec{s}_2\|^2 = d_2^2$$
$$\|\vec{x} - \vec{s}_3\|^2 = d_3^2$$

Now, let's show this algebraically with trilateration. We can expand each left hand side out in terms of the definition of the norm:

$$\|\vec{x} - \vec{s}_i\|^2 = \langle \vec{x} - \vec{s}_i, \vec{x} - \vec{s}_i \rangle = (\vec{x} - \vec{s}_i)^T (\vec{x} - \vec{s}_i)$$

$$\vec{x}^T\vec{x} - 2\vec{x}^T\vec{s}_1 + \vec{s}_1^T\vec{s}_1 = d_1^2$$
$$\vec{x}^T\vec{x} - 2\vec{x}^T\vec{s}_2 + \vec{s}_2^T\vec{s}_2 = d_2^2$$
$$\vec{x}^T\vec{x} - 2\vec{x}^T\vec{s}_3 + \vec{s}_3^T\vec{s}_3 = d_3^2$$

Finally, take one equation and subtract it from the other two to get a system of linear equations in $\vec{x}$:

$$2\vec{x}^T\vec{s}_3 - 2\vec{x}^T\vec{s}_1 = d_1^2 - d_3^2 + \vec{s}_3^T\vec{s}_3 - \vec{s}_1^T\vec{s}_1$$
$$2\vec{x}^T\vec{s}_3 - 2\vec{x}^T\vec{s}_2 = d_2^2 - d_3^2 + \vec{s}_3^T\vec{s}_3 - \vec{s}_2^T\vec{s}_2$$

We can express as a matrix equation in $\vec{x}$:

$$\begin{bmatrix} 2(\vec{s}_3 - \vec{s}_1)^T \\ 2(\vec{s}_3 - \vec{s}_2)^T \end{bmatrix} \vec{x} = \begin{bmatrix} d_1^2 - d_3^2 + \|\vec{s}_3\|^2 - \|\vec{s}_1\|^2 \\ d_2^2 - d_3^2 + \|\vec{s}_3\|^2 - \|\vec{s}_2\|^2 \end{bmatrix}$$

We have that:

$$2(\vec{s}_3 - \vec{s}_1) = \begin{bmatrix} -30 \\ 2 \end{bmatrix}$$
$$2(\vec{s}_3 - \vec{s}_2) = \begin{bmatrix} -24 \\ 14 \end{bmatrix}$$
$$d_1^2 - d_3^2 + \|\vec{s}_3\|^2 - \|\vec{s}_1\|^2 = 25 - 169 + 157 - 41 = -28$$
$$d_2^2 - d_3^2 + \|\vec{s}_3\|^2 - \|\vec{s}_2\|^2 = 4 - 169 + 157 - 2 = -10$$

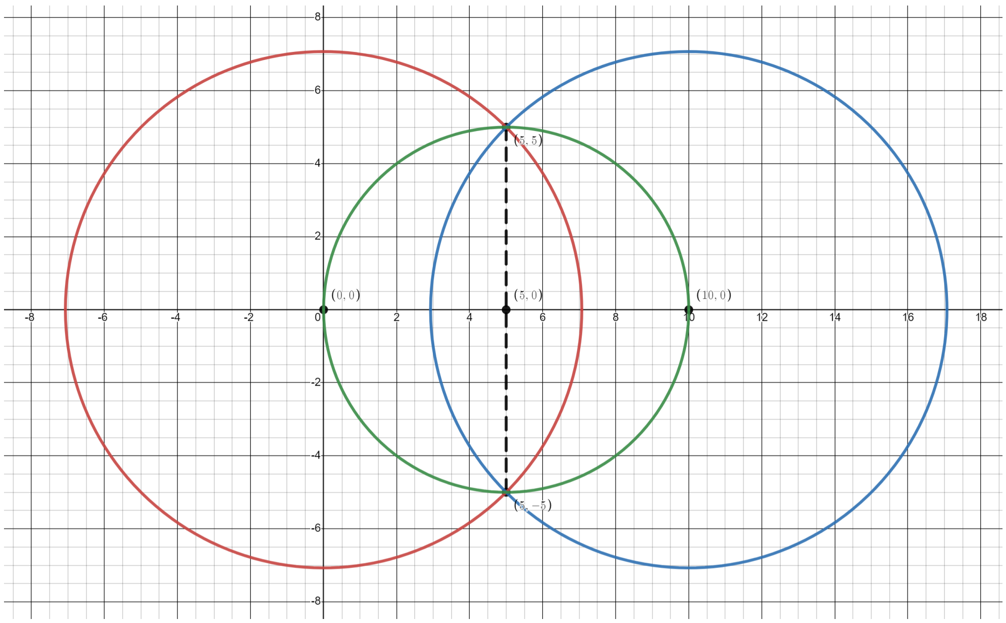
Which gives us the system $\begin{bmatrix} -30 & 2 \\ -24 & 14 \end{bmatrix} \vec{x} = \begin{bmatrix} -28 \\ -10 \end{bmatrix}$ with solution $\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.
A solution existing for this system of linear equations does not necessarily guarantee consistency of the system of nonlinear equations, but we can validate:

$$\left\| \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 4 \\ 5 \end{bmatrix} \right\|^2 = \left\| \begin{bmatrix} -3 \\ -4 \end{bmatrix} \right\|^2 = 25 = d_1^2$$
$$\left\| \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\|^2 = \left\| \begin{bmatrix} 0 \\ 2 \end{bmatrix} \right\|^2 = 4 = d_2^2$$
$$\left\| \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} -11 \\ 6 \end{bmatrix} \right\|^2 = \left\| \begin{bmatrix} 12 \\ -5 \end{bmatrix} \right\|^2 = 169 = d_3^2$$

(b) $\vec{s}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, d_1 = 5\sqrt{2}; \vec{s}_2 = \begin{bmatrix} 10 \\ 0 \end{bmatrix}, d_2 = 5\sqrt{2}; \vec{s}_3 = \begin{bmatrix} 5 \\ 0 \end{bmatrix}, d_3 = 5$

Why can't we precisely determine our location, even though we have the same number of measurements as part (a)? Can we use our original constraints to narrow down our set of possible solutions we got from trilateration?

Solution: Graphing our constraints gives us two points of intersection.



Now, let's try to algebraically solve for these points using trilateration. Using the linearization approach from part (a) we get:

$$2(\vec{s}_3 - \vec{s}_1) = \begin{bmatrix} 10 \\ 0 \end{bmatrix}$$
$$2(\vec{s}_3 - \vec{s}_2) = \begin{bmatrix} -10 \\ 0 \end{bmatrix}$$
$$d_1^2 - d_3^2 + \|\vec{s}_3\|^2 - \|\vec{s}_1\|^2 = 50 - 25 + 25 - 0 = 50$$
$$d_2^2 - d_3^2 + \|\vec{s}_3\|^2 - \|\vec{s}_2\|^2 = 50 - 25 + 25 - 100 = -50$$

Which gives us the system $\begin{bmatrix} 10 & 0 \\ -10 & 0 \end{bmatrix} \vec{x} = \begin{bmatrix} 50 \\ -50 \end{bmatrix}$ with solution $\vec{x} = \begin{bmatrix} 5 \\ \alpha \end{bmatrix}$. We can see that by having collinear beacons, we may not be able to precisely determine our location (short exercise: how does this relate to span and vector spaces?)

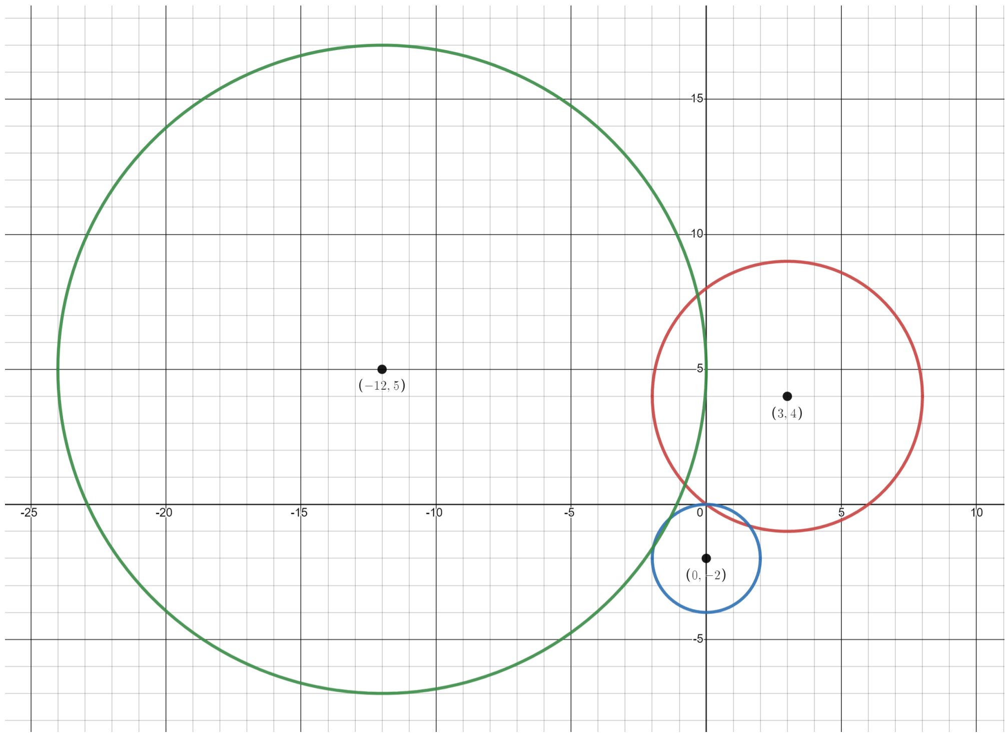
However, from the graph we know that not all values of α are valid, so we can plug our solution back into the third distance equation:

$$\left\| \begin{bmatrix} 5 \\ \alpha \end{bmatrix} - \begin{bmatrix} 5 \\ 0 \end{bmatrix} \right\|^2 = 5^2 \implies \alpha^2 = 25 \implies \alpha = \pm 5$$

The system of nonlinear equations is consistent with this solution. We do not have enough information to uniquely determine our location, but we know we are at either $\vec{x} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$ or $\vec{x} = \begin{bmatrix} 5 \\ -5 \end{bmatrix}$.

(c) $\vec{s}_1 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}, d_1 = 5; \vec{s}_2 = \begin{bmatrix} 0 \\ -2 \end{bmatrix}, d_2 = 2; \vec{s}_3 = \begin{bmatrix} -12 \\ 5 \end{bmatrix}, d_3 = 12$

Solution: Graphing our equations gives us no points of intersection, meaning that there will be no solutions.



Now, let's show this algebraically with trilateration. Using again what was shown in part (a) we have that:

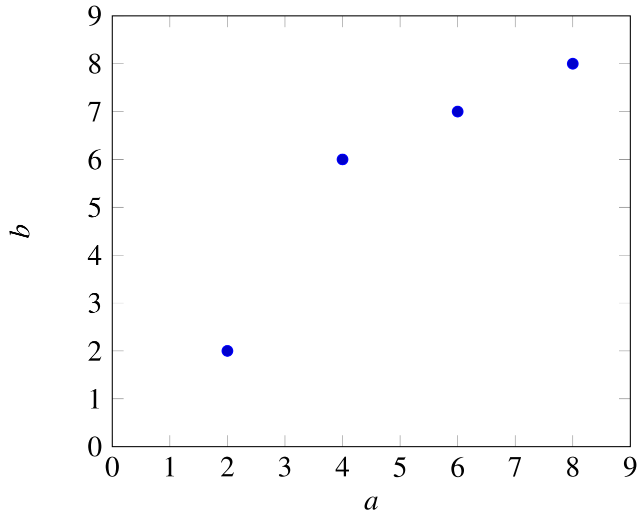
$$2(\vec{s}_3 - \vec{s}_1) = \begin{bmatrix} -30 \\ 2 \end{bmatrix}$$
$$2(\vec{s}_3 - \vec{s}_2) = \begin{bmatrix} -24 \\ 14 \end{bmatrix}$$
$$d_1^2 - d_3^2 + \|\vec{s}_3\|^2 - \|\vec{s}_1\|^2 = 25 - 144 + 169 - 25 = 25$$
$$d_2^2 - d_3^2 + \|\vec{s}_3\|^2 - \|\vec{s}_2\|^2 = 4 - 144 + 169 - 4 = 25$$

Which gives us the system $\begin{bmatrix} -30 & 2 \\ -24 & 14 \end{bmatrix} \vec{x} = \begin{bmatrix} 25 \\ 25 \end{bmatrix}$. While a solution, $\vec{x} = \begin{bmatrix} -\frac{75}{93} \\ \frac{75}{186} \end{bmatrix}$, for this system of linear equations exists, it will yield inconsistent distances when substituted back into the nonlinear equations.

$$\|\vec{s}_1 - x\|^2 = \left(3 + \frac{75}{93}\right)^2 + \left(4 - \frac{75}{186}\right)^2 = 27.43 \neq d_1^2 = 25$$

Therefore there is no solution.

5. Mechanical: Linear Least Squares



a	2	4	6	8
b	2	6	7	8

(a) Consider the above data points. Find the linear model of the form

$$\vec{b} = \vec{a}x$$

that best fits the data, i.e. find the scalar value of $x = \hat{x}$ that minimizes the squared error

$$||\vec{e}||^2 = ||\vec{b} - \vec{a}x||^2 = \left\| \begin{bmatrix} b_1 \\ \vdots \\ b_4 \end{bmatrix} - \begin{bmatrix} a_1 \\ \vdots \\ a_4 \end{bmatrix} x \right\|^2 . \quad (7)$$

Note: By using this linear model, we are implicitly forcing the fit equation to go through the origin. **Do not use IPython for this calculation and show your work.** Once you've computed $\hat{x}$, compute the squared error between your model's prediction and the actual $\vec{b}$ values as shown in Equation 7. Plot the best fit line along with the data points to examine the quality of the fit. (It is okay if your plot of $\vec{b} = \vec{a}x$ is approximate.)

Solution:

Define $\vec{a} = \begin{bmatrix} 2 & 4 & 6 & 8 \end{bmatrix}^T$ and $\vec{b} = \begin{bmatrix} 2 & 6 & 7 & 8 \end{bmatrix}^T$. Applying the linear least squares formula, we get

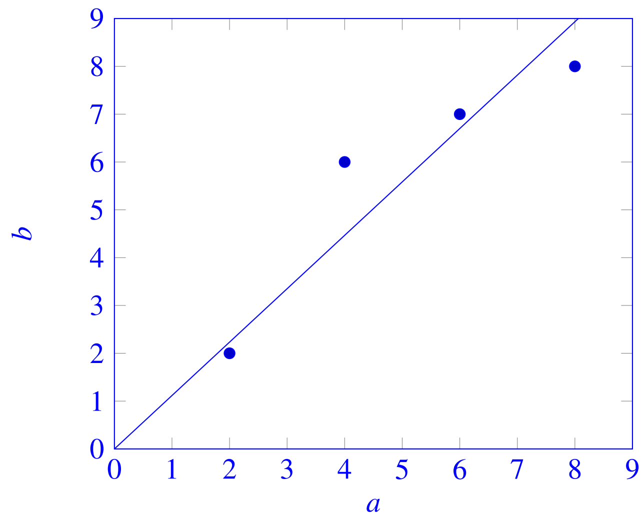
$$\begin{aligned} \hat{x} &= (\vec{a}^T\vec{a})^{-1}\vec{a}^T\vec{b} \\ &= \left(\begin{bmatrix} 2 & 4 & 6 & 8 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ 6 \\ 8 \end{bmatrix} \right)^{-1} \begin{bmatrix} 2 & 4 & 6 & 8 \end{bmatrix} \begin{bmatrix} 2 \\ 6 \\ 7 \\ 8 \end{bmatrix} \\ &= (120)^{-1}(134) = 1.1167 \end{aligned}$$

The error between the model's prediction and actual b values is

$$\begin{aligned} \vec{e} &= \vec{b} - \hat{\vec{b}} = \vec{b} - \hat{x}\vec{a} \\ &= \begin{bmatrix} 2 \\ 6 \\ 7 \\ 8 \end{bmatrix} - 1.1167 \begin{bmatrix} 2 \\ 4 \\ 6 \\ 8 \end{bmatrix} = \begin{bmatrix} -0.234 \\ 1.534 \\ 0.3 \\ -0.934 \end{bmatrix} \end{aligned}$$

and the sum of squared errors is

$$\vec{e}^T\vec{e} = 3.367$$



(b) You will notice from your graph that you can get a better fit by adding a b-intercept. That is we can get a better fit for the data by assuming a linear model of the form

$$\vec{b} = x_1\vec{a} + x_2.$$

In order to do this, we need to augment our $\mathbf{A}$ matrix for the least squares calculation with a column of 1's (do you see why?), so that it has the form

$$\mathbf{A} = \begin{bmatrix} a_1 & 1 \\ \vdots & \vdots \\ a_4 & 1 \end{bmatrix}.$$

Find x_1 and x_2 that minimize the squared error

$$\|\vec{e}\|^2 = \left\|\vec{b} - \mathbf{A}\vec{x}\right\|^2 = \left\| \begin{bmatrix} b_1 \\ \vdots \\ b_4 \end{bmatrix} - \begin{bmatrix} a_1 & 1 \\ \vdots & \vdots \\ a_4 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\|^2. \tag{8}$$

Do not use IPython for this calculation and show your work.

Compute the squared error between your model's prediction and the actual $\vec{b}$ values as shown in Equation 8. Plot your new linear model. Is it a better fit for the data?

Solution:

Let $\vec{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T$. Using the linear least squares formula with the new augmented $\mathbf{A}$ matrix, we calculate the optimal approximation of $\vec{x}$ as

$$\begin{aligned}
\hat{\vec{x}} &= \left(\mathbf{A}^T\mathbf{A}\right)^{-1}\mathbf{A}^T\vec{b} \\
&= \left(\begin{bmatrix}2 & 4 & 6 & 8\\ 1 & 1 & 1 & 1\end{bmatrix}\begin{bmatrix}2 & 1\\ 4 & 1\\ 6 & 1\\ 8 & 1\end{bmatrix}\right)^{-1}\begin{bmatrix}2 & 4 & 6 & 8\\ 1 & 1 & 1 & 1\end{bmatrix}\begin{bmatrix}2\\ 6\\ 7\\ 8\end{bmatrix} \\
&= \begin{bmatrix}120 & 20\\ 20 & 4\end{bmatrix}^{-1}\begin{bmatrix}2 & 4 & 6 & 8\\ 1 & 1 & 1 & 1\end{bmatrix}\begin{bmatrix}2\\ 6\\ 7\\ 8\end{bmatrix} \\
&= \frac{1}{120(4)-20(20)}\begin{bmatrix}4 & -20\\ -20 & 120\end{bmatrix}\begin{bmatrix}134\\ 23\end{bmatrix} \\
\hat{\vec{x}} = \begin{bmatrix}\hat{x}_1\\ \hat{x}_2\end{bmatrix} &= \begin{bmatrix}0.95\\ 1\end{bmatrix}
\end{aligned}$$

The linear model's prediction of $\vec{b}$ is given by

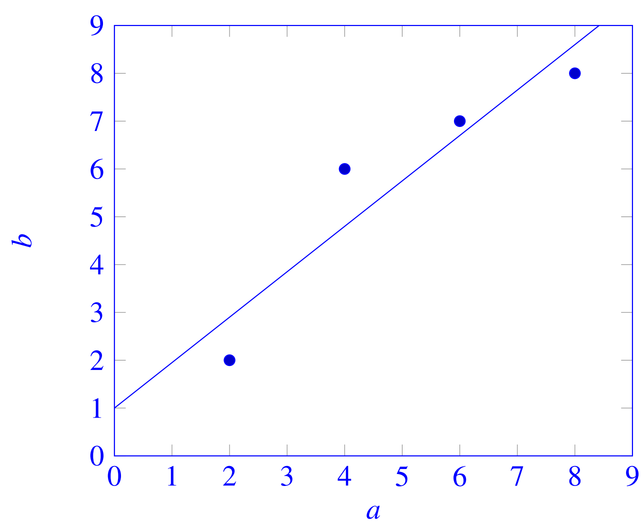
$$\hat{\vec{b}} = \mathbf{A}\hat{\vec{x}} = \begin{bmatrix}2 & 1\\ 4 & 1\\ 6 & 1\\ 8 & 1\end{bmatrix}\begin{bmatrix}0.95\\ 1\end{bmatrix} = \begin{bmatrix}2.9\\ 4.8\\ 6.7\\ 8.6\end{bmatrix}$$

and the error is given by

$$\vec{e} = \vec{b} - \hat{\vec{b}} = \begin{bmatrix}-0.9 & 1.2 & 0.3 & -0.6\end{bmatrix}^T$$

The summed squared error is

$$\|\vec{e}\|^2 = \vec{e}^T\vec{e} = 2.7$$



We can see both qualitatively from the plots and quantitatively from the sum of the squared errors that the fit is better with the b-intercept.

6. Proof: Least Squares

Let $\vec{\hat{x}}$ be the solution to a linear least squares problem.

$$\vec{\hat{x}} = \underset{\vec{x}}{\operatorname{argmin}} \left\| \vec{b} - \mathbf{A}\vec{x} \right\|^2$$

Show that the minimizing least squares error vector $\vec{e} = \vec{b} - \mathbf{A}\vec{\hat{x}}$ is orthogonal to the columns of $\mathbf{A}$ by direct manipulation (i.e. plug the formula for the linear least squares solution $\vec{\hat{x}}$ into the error vector and then check if $\mathbf{A}^T\vec{e} = \vec{0}$.)

Solution:

We want to show that the error in the linear least squares estimate is orthogonal to the columns of the $\mathbf{A}$, i.e., we want to show that $\mathbf{A}^T\vec{e} = \mathbf{A}^T(\vec{b} - \mathbf{A}\vec{\hat{x}})$ is the zero vector. Plugging in the linear least squares formula for $\vec{\hat{x}}$, we get

$$\begin{aligned}
\mathbf{A}^T\vec{e} = \mathbf{A}^T\left(\vec{b} - \mathbf{A}\vec{\hat{x}}\right) &= \mathbf{A}^T\left(\vec{b} - \mathbf{A}\left(\mathbf{A}^T\mathbf{A}\right)^{-1}\mathbf{A}^T\vec{b}\right) \\
&= \mathbf{A}^T\vec{b} - \mathbf{A}^T\mathbf{A}\left(\mathbf{A}^T\mathbf{A}\right)^{-1}\mathbf{A}^T\vec{b} \\
&= \mathbf{A}^T\vec{b} - \mathbf{I}\mathbf{A}^T\vec{b} \\
&= \mathbf{A}^T\vec{b} - \mathbf{A}^T\vec{b} = \vec{0}
\end{aligned}$$

7. Trilateration With Noise!

Learning Goal: *This problem will help to understand how noise affects the accuracy of trilateration and consistency of the corresponding system of equations.*

In this question, we will explore how various types of noise affect the quality of triangulating a point on the 2D plane to see when trilateration works well and when it does not.

First, we will remind ourselves of the fundamental equations underlying trilateration.

(a) There are four beacons at the known coordinates $(x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4)$. You are located at some unknown coordinate (x, y) that you want to determine. The distance between your location and each of the four beacons are d_1 through d_4, respectively. Write down one equation for each beacon that relates the coordinates to the distances using the Pythagorean Theorem.

Solution: For each beacon, we have the equation

$$(x - x_i)^2 + (y - y_i)^2 = d_i^2,$$

for $i \in \{1, 2, 3, 4\}$.

(b) Unfortunately, the above system of equations is nonlinear, so we can't use least squares or Gaussian Elimination to solve it. We will use the technique discussed in lecture to obtain a system of linear equations. In particular, we can subtract the first of the above equations (involving x_1, y_1 and d_1) from the other three to obtain three linear equations (cancel out the nonlinear terms). Write down these three linear equations.

Combine the three equations in the above system into a single matrix equation of the form

$$\mathbf{A}\begin{bmatrix} x \\ y \end{bmatrix} = \vec{b}.$$

Solution: Subtracting the 1st equation from the ith, we obtain

$$(x-x_i)^2-(x-x_1)^2+(y-y_i)^2-(y-y_1)^2=d_i^2-d_1^2,$$

so expanding and canceling the x^2 and y^2 terms, we obtain

$$-2xx_i+x_i^2+2xx_1-x_1^2-2yy_i+y_i^2+2yy_1-y_1^2=d_i^2-d_1^2$$

for $i \in \{2,3,4\}$.
Rearranging each of the above equations, we obtain

$$(-2x_i+2x_1)x+(-2y_i+2y_1)y=(d_i^2-x_i^2-y_i^2)-(d_1^2-x_1^2-y_1^2)$$

for $i \in \{2,3,4\}$. Stacking and writing in matrix form, we obtain

$$\begin{bmatrix} 2(-x_2+x_1) & 2(-y_2+y_1) \\ 2(-x_3+x_1) & 2(-y_3+y_1) \\ 2(-x_4+x_1) & 2(-y_4+y_1) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} (d_2^2-x_2^2-y_2^2)-(d_1^2-x_1^2-y_1^2) \\ (d_3^2-x_3^2-y_3^2)-(d_1^2-x_1^2-y_1^2) \\ (d_4^2-x_4^2-y_4^2)-(d_1^2-x_1^2-y_1^2) \end{bmatrix}.$$

(c) Now, go to the IPython notebook. In the notebook we are given three possible sets of measurements for the distances of each beacon from the receiver:

i. `ideal_distances`: the ideal set of measurements, the true distances of our receiver to the beacons. $d_1=d_2=d_3=d_4=5$.

ii. `imperfect_distances`: imperfect measurements. $d_1=5.5, d_2=4.5, d_3=5, d_4=5$.

iii. `one_bad_distances`: mostly perfect measurements, but d_1 is a very bad measurement. $d_1 = 7.5$ and $d_2=d_3=d_4=5$.

Plot the graph illustrating the case when the receiver has received `ideal_distances` and visually solve for the position of the observer (x,y). What is the coordinate?

Solution: From the plot, it is clear that $(x,y)=(0,0)$, since all four circles intersect at that point.

(d) You will now set up the above linear system using IPython. Fill in each element of the matrix $\mathbf{A}$ that you found in part (c).

Solution: $\mathbf{A} = \begin{bmatrix} 2(-x_2+x_1) & 2(-y_2+y_1) \\ 2(-x_3+x_1) & 2(-y_3+y_1) \\ 2(-x_4+x_1) & 2(-y_4+y_1) \end{bmatrix}.$

See the IPython notebook for the actual code.

(e) Similarly, fill in the entries of $\vec{b}$ from part (c) in the `make_b` function.

Solution: $\vec{b} = \begin{bmatrix} (d_2^2-x_2^2-y_2^2)-(d_1^2-x_1^2-y_1^2) \\ (d_3^2-x_3^2-y_3^2)-(d_1^2-x_1^2-y_1^2) \\ (d_4^2-x_4^2-y_4^2)-(d_1^2-x_1^2-y_1^2) \end{bmatrix}.$

See the IPython notebook for the actual code.

(f) Now, you should be able to plot the estimated position of (x,y) using the supplied code for the `ideal_distances` observations. Modify the code to estimate (x,y) for `imperfect_distances` and `one_bad_distances`, and comment on the results.

In particular, for `one_bad_distances` would you intuitively have chosen the same point that our trilateration solution did knowing that only one measurement was bad?

Solution: We see that the solution to (x,y) moves away from the origin in the latter two cases. For `one_bad_distances`, even through three out of the four circles intersect at the origin (suggesting that $(x,y)=(0,0)$), our least squares approach picks a point away from the origin, indicating that it might not be determining the best solution possible.

(g) We define the "cost" of a position (x,y) to be the sum of the squares of the differences in distance of that position from the observation, as defined symbolically in the notebook. Study the heatmap of the cost of various positions on the plane, and make sure you see why $(0,0)$ appears to be the point with the lowest cost.

Now, compare the cost of $(0,0)$ with the cost of your estimated position obtained from the least-squares solution in all three cases. For which cases does least squares do worse?

Solution: See IPython solutions. For `ideal_distances`, both approaches yield a cost of 0.0. For `imperfect_distances`, $(0,0)$ is actually slightly worse than our least-squares solution, but in both cases the costs are fairly low.

For `one_bad_distances`, the costs are lower at $(0,0)$ compared to the least-squares solution, as we expected from the heatmap.

8. Homework Process and Study Group

Who did you work with on this homework? List names and student ID's. (In case you met people at homework party or in office hours, you can also just describe the group.) How did you work on this homework? If you worked in your study group, explain what role each student played for the meetings this week.

Solution:

I first worked by myself for 2 hours, but got stuck on problem 5. Then I met with my study group.

XYZ played the role of facilitator ... etc. We were still stuck on problem 5 so we went to office hours to talk about the problem.

Then I went to homework party for a few hours, where I finished the homework.