EECS 16A Designing Information Devices and Systems I
Fall 2020 Homework 2

This homework is due September 11, 2020, at 23:59.
Self-grades are due September 14, 2020, at 23:59.

Submission Format
Your homework submission should consist of a single PDF file that contains all of your answers (any handwritten answers should be scanned) as well as your IPython notebook saved as a PDF.

If you do not attach a PDF "printout" of your IPython notebook, you will not receive credit for problems that involve coding. Make sure that your results and your plots are visible. Assign the IPython printout to the correct problem(s) on Gradescope.

1. Reading Assignment

For this homework, please read the rest of Note 1 and Note 2A (or Note 1B and Note 2A). The notes 1/1B and 2A will provide an overview of systems of linear equations, vectors, and matrices. You are always welcome and encouraged to read beyond this as well, in particular, a quick look at Note 3 will help you. Describe how Gaussian Elimination can help you understand if there are no solutions to a particular system of equations? What about a unique solution? Does a row of zeros always mean there are infinite solutions?

Solution: This is an example solution and you should give yourself full credit for any reasonable answers.

There are three situations that can result following Gaussian elimination. We assume that the system of equations has n unknowns.

Case 1: No Solution
If the augmented matrix has any rows with all-zero variable coefficients but a nonzero result (corresponding to $0 = a$ where $a \neq 0$), then there is no solution.

Case 2: Infinite Solutions
If the augmented matrix has fewer than n non-zero rows (i.e. fewer than n entries in pivot position) and any rows with all-zero variable coefficients also have a zero result (corresponding to $0 = 0$), then there are infinite solutions.

Case 3: Unique Solution
If the augmented matrix has n non-zero rows (i.e. n entries in pivot position) and any rows with all-zero variable coefficients also have a zero result (corresponding to $0 = 0$), then there is a unique solution.

2. Filtering Out The Troll

Solution: `#SystemsOfEquations #LinearCombination`

Learning Goal: *The goal of this problem is to represent a practical scenario using a simple model of directional microphones. Students will tackle the problem of sound reconstruction through solving a system of linear equations.*

You attended a very important public speech and recorded it using a recording device that had two directional microphones. However, there was a person in the audience who was trolling around, adding interference to the recording. When you went back home to listen to the recording, you realized that the two recordings were dominated by the troll's interference and you could not hear the speech. Fortunately, since your recording device contained two microphones, you realized there is a way to combine the two individual microphone recordings so that the troll's interference is removed. You remembered the locations of the speaker and the troll and created the diagram shown in Figure 1. You (and your two microphones) are located at the origin.

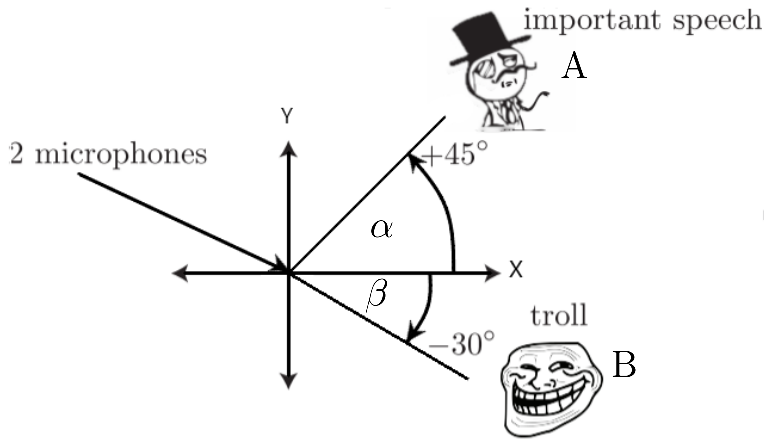


Figure 1: Locations of the speaker and the troll.

Each directional microphone records signals differently based on their angles of arrival. The first microphone weights (multiplies) a signal, coming from an angle θ with respect to the x-axis, by the factor $f_1(\theta) = \cos(\theta)$. If two signals are simultaneously playing (as is the case with the speech and the troll interference), then a linear combination (i.e. a weighted sum) of both the signals is recorded, each weighted by the respective $f_1(\theta)$ for their angles. The second microphone weights a signal, coming from an angle θ with respect to the x-axis, by the factor $f_2(\theta) = \sin(\theta)$. Again, if there are two simultaneous signals, then the second microphone also records the weighted sum of both the signals.

For example, an audio source that lies on the x axis will be recorded by the first microphone with weight equal to 1 (since $\cos(0) = 1$), but will not be recorded up by the second microphone (since $\sin(0) = 0$). Note that the weights can also be negative.

Let us represent the speech sample at a particular time-instant by the variable a and the interference caused by the troll at the same time-instant by the variable b. Remember, we do not know either a or b. The recording of the first microphone at that time instant is given by m_1:

$$m_1 = f_1(\alpha) \cdot a + f_1(\beta) \cdot b,$$

and the second microphone recorded the signal

$$m_2 = f_2(\alpha) \cdot a + f_2(\beta) \cdot b.$$

where α and β are the angles at which the public speaker A and the troll B respectively are located with respect to the x-axis, and vectors a and b are the audio signals produced by the public speaker A and the troll B respectively.

(As a side note, we could represent the entire speech with a vector $\vec{a}$ by stacking all the speech samples on top of each other, and this is what we would typically do in a real-world speech processing example. However, here we consider just one time instant of the speech for simplicity.)

(a) Plug in the values of α and β to write the recordings of the two microphones m_1 and m_2 as a linear combination (i.e. a weighted sum) of a and b.

Solution:

$$\begin{aligned} m_1 &= \cos\left(\frac{\pi}{4}\right)\cdot a + \cos\left(-\frac{\pi}{6}\right)\cdot b \\ &= \frac{1}{\sqrt{2}}\cdot a + \frac{\sqrt{3}}{2}\cdot b \\ m_2 &= \sin\left(\frac{\pi}{4}\right)\cdot a + \sin\left(-\frac{\pi}{6}\right)\cdot b \\ &= \frac{1}{\sqrt{2}}\cdot a - \frac{1}{2}\cdot b \end{aligned}$$

(b) Solve the system you wrote out on the earlier part to recover the important speech a, as a weighted combination of m_1 and m_2. In other words, write $a = u\cdot m_1 + v\cdot m_2$ (where u and v are scalars). What are the values of u and v?

Solution: Solving the system of linear equations yields

$$a = \frac{\sqrt{2}}{1+\sqrt{3}}\cdot\left(m_1 + \sqrt{3}m_2\right).$$

Therefore, the values are $u = \frac{\sqrt{2}}{1+\sqrt{3}}$ and $v = \frac{\sqrt{6}}{1+\sqrt{3}}$.

It is fine if you solved this either using IPython or by hand using any valid technique. The easiest approach is to subtract either of the two equations from the other and immediately see that $b = \frac{2}{\sqrt{3}+1}(m_1 - m_2)$. Substituting b back into the second equation and multiplying through by $\sqrt{2}$ gives that $a = \sqrt{2}(m_2 + \frac{1}{\sqrt{3}+1}(m_1 - m_2))$, which simplifies to the expression given above.

Notice that subtracting one equation from the other is natural given the symmetry of the microphone patterns and the fact that the patterns intersect at the 45 degree line where the important speech is happening, and the fact that $\sin(45°) = \cos(45°)$. So we know that the result of subtracting one microphone recording from the other results in only the trolls contribution. Once we have the troll contribution, we can remove it and obtain the important speakers sole content.

(c) Partial IPython code can be found in `prob2.ipynb`, which you can access through the datahub link associated with this assignment on the course website. Complete the code to get the signal of the important speech. Write out what the speaker says. (Optional: Where is the speech taken from?)

Note: You may have noticed that the recordings of the two microphones sound remarkably similar. This means that you could recover the real speech from two "trolled" recordings that sound almost identical! Leave out the fact that the recordings are actually different, and have some fun with your friends who aren't lucky enough to be taking EECS16A.

Solution:

The solution code can be found in `sol2.ipynb`. The speaker says: "All human beings are born free and equal in dignity and rights." and the speech was taken from the Universal Declaration of Human Rights.

The idea of using multiple microphones to isolate speech is interesting and is increasingly used in practice. Furthermore, similar techniques are used in wireless communication both by cellular systems like LTE and increasingly by WiFi hotspots. (This is why they often have multiple antennas).

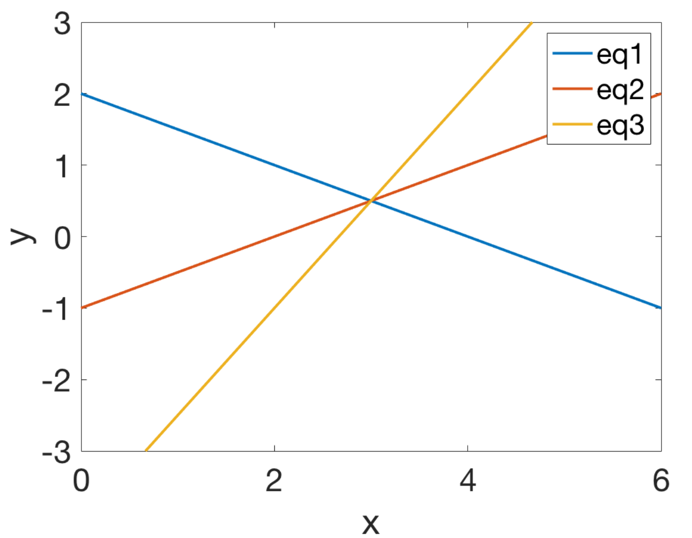
3. Gaussian Elimination

(a) In this problem we will investigate the relationship between Gaussian elimination and the geometric interpretation of linear equations. You are welcome to draw plots by hand or using software. Please be sure to label your equations with a legend on the plot.

i. Draw the following set of linear equations in the x-y plane. If the lines intersect, write down the point or points of intersection.

$$x + 2y = 4 \tag{1}$$
$$2x - 4y = 4 \tag{2}$$
$$3x - 2y = 8 \tag{3}$$

Solution:
The three lines intersect at the point (3,0.5).



ii. Write the above set of linear equations in augmented matrix form and do the first step of Gaussian elimination to eliminate the x variable from equation 2. Now, the second row of the augmented matrix has changed. Plot the corresponding new equation created in this step on the same graph as above. What do you notice about the new line you draw?

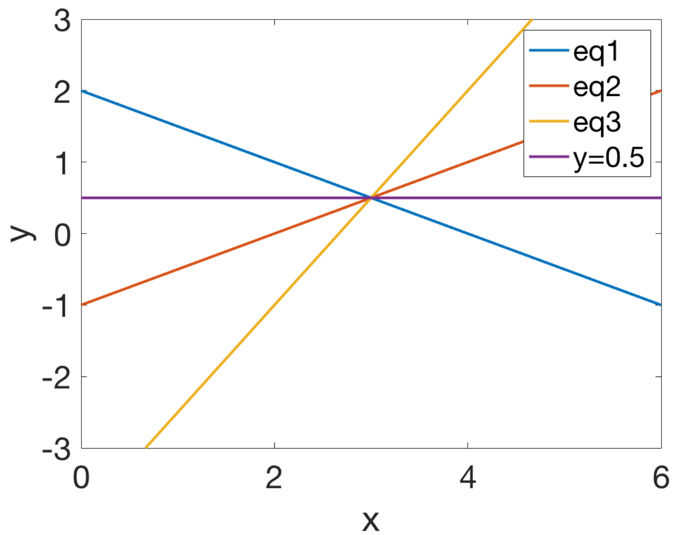
Solution: We start with the following augmented matrix:

$$\left[\begin{array}{cc|c} 1 & 2 & 4 \\ 2 & -4 & 4 \\ 3 & -2 & 8 \end{array}\right]$$

We then eliminate x from the second equation by subtracting 2×Row 1 from Row 2:

$$\text{Row 2: subtract } 2\times\text{Row 1} \implies \left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & -8 & -4 \\ 3 & -2 & 8 \end{array}\right]$$

So equation 2 becomes $-8y = -4$, which is equivalent to $y = 0.5$. You will notice that the line $y = 0.5$ intersects with the three lines you drew previously.



iii. Complete all of the steps of Gaussian elimination including back substitution. Now plot the new equations represented by the rows of the augmented matrix in the last step (after completing back substitution) on the same graph as above. What do you notice about the new line you draw?

Solution:

We continue from the previous part, where we had the following augmented matrix:

$$\left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & -8 & -4 \\ 3 & -2 & 8 \end{array}\right]$$

and take the following steps to complete Gaussian elimination:

$$\text{Row 3: subtract } 3\times\text{Row 1} \quad \Longrightarrow \quad \left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & -8 & -4 \\ 0 & -8 & -4 \end{array}\right]$$

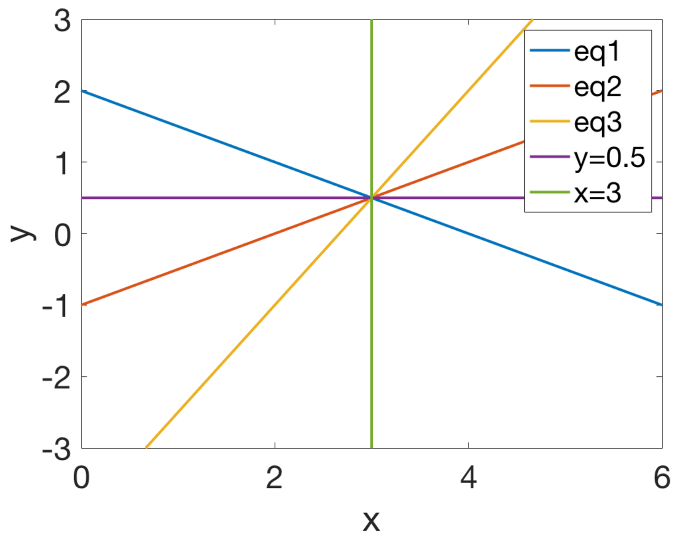
$$\text{Row 2: divide by } -8 \quad \Longrightarrow \quad \left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & 1 & 0.5 \\ 0 & -8 & -4 \end{array}\right]$$

$$\text{Row 3: subtract } -8\times\text{Row 2} \quad \Longrightarrow \quad \left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & 1 & 0.5 \\ 0 & 0 & 0 \end{array}\right]$$

$$\text{Row 1: subtract } 2\times\text{Row 2} \quad \Longrightarrow \quad \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 0.5 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore, we end up with the solution $x = 3$ and $y = 0.5$.

Plotting the new equation $x = 3$ on the same graph as before, we see that that all five lines intersect at the same point (3,0.5).



(b) Write the following set of linear equations in augmented matrix form and use Gaussian elimination to determine if there are no solutions, infinite solutions, or a unique solution. If any solutions exist, determine what they are. You may do this problem by hand or use a computer. We encourage you to try it by hand to ensure you understand Gaussian elimination.

$$\begin{aligned} x+2y+5z &= 3 \\ x+12y+6z &= 1 \\ 2y+z &= 4 \\ 3x+16y+16z &= 7 \end{aligned}$$

Solution:

Writing the system in augmented matrix form we get the following:

$$\left[\begin{array}{ccc|c} 1 & 2 & 5 & 3 \\ 1 & 12 & 6 & 1 \\ 0 & 2 & 1 & 4 \\ 3 & 16 & 16 & 7 \end{array}\right]$$

We eliminate the x variables from the second and fourth equations:

$$\begin{array}{c} \text{Row 2: subtract Row 1} \\ \text{Row 4: subtract } 3\times\text{Row 1} \end{array} \implies \left[\begin{array}{ccc|c} 1 & 2 & 5 & 3 \\ 0 & 10 & 1 & -2 \\ 0 & 2 & 1 & 4 \\ 0 & 10 & 1 & -2 \end{array}\right]$$

We then divide Row 2 by 10 to get a 1 in the pivot position:

$$\text{Row 2: divide by 10} \implies \left[\begin{array}{ccc|c} 1 & 2 & 5 & 3 \\ 0 & 1 & 0.1 & -0.2 \\ 0 & 2 & 1 & 4 \\ 0 & 10 & 1 & -2 \end{array}\right]$$

Next, we eliminate the y variables from the third and fourth equations:

$$\begin{array}{c}\text{Row 3: subtract } 2\times\text{Row 2}\\ \text{Row 4: subtract } 10\times\text{Row 2}\end{array} \implies \left[\begin{array}{ccc|c} 1 & 2 & 5 & 3 \\ 0 & 1 & 0.1 & -0.2 \\ 0 & 0 & 0.8 & 4.4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

We divide Row 3 by 0.8 to get a 1 in the pivot position:

$$\text{Row 3: divide by } 0.8 \implies \left[\begin{array}{ccc|c} 1 & 2 & 5 & 3 \\ 0 & 1 & 0.1 & -0.2 \\ 0 & 0 & 1 & 5.5 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

We then proceed with back-substitution:

$$\begin{array}{c}\text{Row 2: subtract } 0.1\times\text{Row 3}\\ \text{Row 1: subtract } 5\times\text{Row 3}\end{array} \implies \left[\begin{array}{ccc|c} 1 & 2 & 0 & -24.5 \\ 0 & 1 & 0 & -0.75 \\ 0 & 0 & 1 & 5.5 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\text{Row 1: subtract } 2\times\text{Row 2} \implies \left[\begin{array}{ccc|c} 1 & 0 & 0 & -23 \\ 0 & 1 & 0 & -0.75 \\ 0 & 0 & 1 & 5.5 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This final matrix is in reduced row echelon form. The first three rows of the matrix have non-zero elements in pivot position, for a system with three unknowns, and the fourth row is a row of zeros, so we can conclude there is a unique solution: $x = -23$, $y = -0.75$, and $z = 5.5$.

(c) Consider the following system of equations:

$$\begin{aligned} x + 2y + 5z &= 6 \\ 3x + 9y + 6z &= 3 \end{aligned}$$

You are given a set S of candidate solutions,

$$S = \left\{ \vec{v} \;\middle|\; \vec{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 16 \\ -5 \\ 0 \end{bmatrix} + \begin{bmatrix} -11 \\ 3 \\ 1 \end{bmatrix} t \quad , \quad t \in \mathbb{R} \right\}$$

This vector notation can be expressed in terms of its components:

$$\vec{v} = \begin{bmatrix} 16 - 11t \\ -5 + 3t \\ t \end{bmatrix} \quad \text{means} \quad \begin{aligned} x &= 16 - 11t \\ y &= -5 + 3t \\ z &= t \end{aligned}$$

Show, by substitution, that any $\vec{v} \in S$ is a solution to the system of equations given above. Note that this means that the candidate solution must satisfy the system of equations for all $t \in \mathbb{R}$.

Solution:

We can use direct substitution to show any $\vec{v} \in S$ is a solution to the equations.

$$\vec{v} = \begin{bmatrix} 16-11t \\ -5+3t \\ t \end{bmatrix} \quad \Longrightarrow \quad \begin{aligned} x &= 16-11t \\ y &= -5+3t \\ z &= t \end{aligned}$$

We now take these expressions for x, y, and z, and show that the system of equations is satisfied for all $t \in \mathbb{R}$.

Plugging the expressions into equation 1 yields the following:

$$\begin{aligned} (16-11t)+2(-5+3t)+5t &= 6 \\ 16-11t-10+6t+5t &= 6 \\ 6 &= 6 \end{aligned}$$

This holds for all $t \in \mathbb{R}$.

Similarly, plugging the expressions into equation 2 yields the following:

$$\begin{aligned} 3(16-11t)+9(-5+3t)+6t &= 3 \\ 48-33t-45+27t+6t &= 3 \\ 3 &= 3 \end{aligned}$$

This also holds for all $t \in \mathbb{R}$.

Therefore, the candidate solution satisfies the system of equations for all $t \in \mathbb{R}$.

(d) Consider the following system:

$$\begin{aligned} 4x+4y+4z+w+v &= 1 \\ x+y+2z+4w+v &= 2 \\ 5x+5y+5z+w+v &= 0 \end{aligned}$$

If you were to write the above equations in augmented matrix form and use Gaussian elimination to solve the system, you would get the following (for extra practice, you can try and do this yourself):

$$\left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 3 & 16 \\ 0 & 0 & 1 & 0 & -3 & -17 \\ 0 & 0 & 0 & 1 & 1 & 5 \end{array}\right]$$

How many variables are free variables? Determine the solutions to the set of equations.

Solution:

We first note that the given augmented matrix is in reduced row echelon form, which makes sense as it is the final output of the Gaussian elimination algorithm. We observe that the second and fifth columns do not have 1s in pivot position so there are two free variables corresponding to y and v.

Let $y = s$ and let $v = t$, where $s \in \mathbb{R}$ and $t \in \mathbb{R}$.
Using back substitution, we can solve for x, y, z, w, and v in terms of s and t:

$$\text{Row 1:} \quad x + y + 3v = 16 \implies x = 16 - 3t - s$$
$$\text{Row 2:} \quad z - 3v = -17 \implies z = -17 + 3t$$
$$\text{Row 3:} \quad w + v = 5 \implies w = 5 - t$$

The solutions to the system of equations are therefore:

$$x = 16 - 3t - s$$
$$y = s$$
$$z = -17 + 3t$$
$$w = 5 - t$$
$$v = t$$

4. Tyler's Optimal Tea

Solution: `#SystemsOfEquations #GaussianElimination`

Learning Goal: *Recognize a problem that can be cast as a system of linear equations.*

Tyler's Optimal Tea has a unique way of serving its customers. To ensure the best customer experience, each customer gets a combination drink personalized to their tastes. Tyler knows that a lot of customers don't know what they want, so when customers walk up to the counter, they are asked to taste four standard combination drinks that each contain a different mixture of the available pure teas.

Each combination drink (Classic, Roasted, Mountain, and Okinawa) is made of a mixture of pure teas (Black, Oolong, Green, and Earl Grey), with the total amount of pure tea in each combination drink always the same, and equal to one cup. The table below shows the quantity of each pure tea (Black, Oolong, Green, and Earl Grey) contained in each of the four standard combination drinks (Classic, Roasted, Mountain, and Okinawa).

Tea [cups]	Classic	Roasted	Mountain	Okinawa
Black	$\frac{1}{3}$	$\frac{1}{3}$	0	$\frac{2}{3}$
Oolong	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{2}{5}$	$\frac{1}{3}$
Green	0	$\frac{1}{3}$	$\frac{3}{5}$	0
Earl Grey	$\frac{1}{3}$	0	0	0

Initially, the customer's ratings for each of the pure teas are unknown. Tyler's goal is to determine how much the customer likes each of the pure teas, so that an optimal combination drink can then be made. By letting the customer taste and score each of the four standard combination drinks, Tyler can use linear algebra to determine the customer's initially unknown ratings for each of the pure teas. After a customer gives a score (all of the scores are real numbers) for each of the four standard combination drinks, Tyler then calculates how much the customer likes each pure tea and mixes up a special combination drink that will maximize the customer's score.

The score that a customer gives for a combination drink is a linear combination of the ratings of the constituent pure teas, based on their proportion. For example, if a customer's rating for black tea is 6 and oolong tea is 3, then the total score for the Okinawa Tea drink would be $6 \cdot \frac{2}{3} + 3 \cdot \frac{1}{3} = 5$ because Okinawa has $\frac{2}{3}$ black tea and $\frac{1}{3}$ Oolong tea.

Professor Ranade was thirsty after giving the first lecture, so Professor Ranade decided to take a drink break at Tyler's Optimal Tea. Professor Ranade walked in and gave the following ratings:

Combination Drink	**Score**
Classic	7
Roasted	7
Mountain	$7\frac{2}{5}$
Okinawa	$6\frac{1}{3}$

(a) What were Professor Ranade's ratings for each tea? **Work this problem out by hand in terms of the steps. You may use a calculator to do algebra.**

Solution:

Using Professor Ranade's ratings, Tyler mentally records the following system of equations. Let x_b be the customer's rating of black tea, x_o be the customer's rating of oolong tea, x_g be the customer's rating of green tea, and x_e be the customer's rating of earl grey tea.

$$\begin{aligned}
\text{Classic:} \quad & 7 = \frac{1}{3}x_\text{b} + \frac{1}{3}x_\text{o} + \frac{1}{3}x_\text{e} \\
\text{Roasted:} \quad & 7 = \frac{1}{3}x_\text{b} + \frac{1}{3}x_\text{o} + \frac{1}{3}x_\text{g} \\
\text{Mountain:} \quad & 7\frac{2}{5} = \frac{2}{5}x_\text{o} + \frac{3}{5}x_\text{g} \\
\text{Okinawa:} \quad & 6\frac{1}{3} = \frac{2}{3}x_\text{b} + \frac{1}{3}x_\text{o}
\end{aligned}$$

Multiply each equation by the denominator of the fraction (in order to make them easier to read):

$$\begin{aligned}
21 &= x_\text{b} + x_\text{o} + x_\text{e} \\
21 &= x_\text{b} + x_\text{o} + x_\text{g} \\
37 &= 2x_\text{o} + 3x_\text{g} \\
19 &= 2x_\text{b} + x_\text{o}
\end{aligned}$$

The above equations can be written as an augmented matrix:

$$\left[\begin{array}{cccc|c}
1 & 1 & 0 & 1 & 21 \\
1 & 1 & 1 & 0 & 21 \\
0 & 2 & 3 & 0 & 37 \\
2 & 1 & 0 & 0 & 19
\end{array}\right].$$

Row reduce the matrix into reduced row echelon form as follows. (It's fine if you solved the system of equations by hand a different way. Here, however, we will demonstrate how to do it using Gaussian elimination.)

Noting that there is a 1 in the upper left hand corner, subtract Row 1 from Row 2 and 2×Row 1 from Row 4.

$$\begin{array}{c} \text{Row 2: subtract Row 1} \\ \text{Row 4: subtract } 2\times\text{Row 1} \end{array} \implies \left[\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 21 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 2 & 3 & 0 & 37 \\ 0 & -1 & 0 & -2 & -23 \end{array}\right]$$

Since Row 2 has a 0 in the diagonal element, multiply Row 4 by -1 and then switch Rows 2 and 4.

$$\begin{array}{c} \text{Multiply Row 4 by } -1 \\ \text{Switch Row 2 and Row 4} \end{array} \implies \left[\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 21 \\ 0 & 1 & 0 & 2 & 23 \\ 0 & 2 & 3 & 0 & 37 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right]$$

Subtract Row 2 from Row 1 and 2×Row 2 from Row 3.

$$\begin{array}{c} \text{Row 1: subtract Row 2} \\ \text{Row 3: subtract } 2\times\text{Row 2} \end{array} \implies \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & -2 \\ 0 & 1 & 0 & 2 & 23 \\ 0 & 0 & 3 & -4 & -9 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right]$$

Switch Row 3 and Row 4 and subtract 3× the new Row 3 from the new Row 4.

$$\begin{array}{c} \text{Switch Row 3 and Row 4} \\ \text{Row 4: subtract } 3\times\text{Row 3} \end{array} \implies \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & -2 \\ 0 & 1 & 0 & 2 & 23 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 & -9 \end{array}\right]$$

Finally, multiply Row 4 by -1 and add Row 4 to Row 1 and Row 3 and subtract 2×Row 4 from Row 2.

$$\begin{array}{c} \text{Multiply Row 4 by } -1 \\ \text{Row 1, Row 3: add Row 4} \\ \text{Row 2: subtract } 2\times\text{Row 4} \end{array} \implies \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 7 \\ 0 & 1 & 0 & 0 & 5 \\ 0 & 0 & 1 & 0 & 9 \\ 0 & 0 & 0 & 1 & 9 \end{array}\right]$$

Professor Ranade's ratings for each tea are

Tea	Score
Black	7
Oolong	5
Green	9
Earl Grey	9

(b) What mystery tea combination could Tyler put in Professor Ranade's personalized drink to maximize the customer's score? If there is more than one correct answer, state that there are many answers, and give one such combination. What score would Professor Ranade give for the answer you wrote down? Assume the total amount of tea must be one cup.

Solution:

There are many answers. *Any* combination of green tea and earl grey is acceptable as they have equal ratings. More precisely, for any $0 \le a \le 1$, a combination with a cups of green tea and $1-a$ cups of earl grey will yield a score of $9a + 9(1-a) = 9$.

As an example, Tyler could choose $a = \frac{1}{2}$ so that Professor Ranade's drink has $\frac{1}{2}$ cup of green tea and $\frac{1}{2}$ cup of earl grey.

How to see this? Green tea and earl grey are tied for Professor Ranade's favorite tea, so it doesn't make a difference if Tyler substitutes one for the other in any quantity - the score remains 9. It also doesn't make sense to substitute a less preferred tea like black tea for Professor Ranade's favorite tea, or to add a less preferred tea at the expense of the most preferred teas, as this will lead to scores less than 9.

5. Fountain Codes

Learning Goal: *Linear algebra shows up in many important engineering applications. Wireless communication and information theory heavily rely on principles of linear algebra. This problem illustrates some of the techniques used in wireless communication.*

Alice wants to send a message to her friend Bob. Alice sends her message $\vec{m}$ across a wireless channel in the form of a transmission vector $\vec{w}$. Bob receives a vector of symbols denoted as $\vec{r}$.

Alice knows some of the symbols in the transmission vector that she sends may be corrupted, so she needs a way to protect her message from the corruptions. (Transmission corruptions occur commonly in real wireless communication systems, for instance, when your laptop connects to a WiFi router, or your cellphone connects to the nearest cell tower.) Ideally, Alice can come up with a transmission vector such that if some of the symbols get corrupted, Bob can still figure out what Alice is trying to say!

One way to accomplish this goal is to use fountain codes, which are part of a broader family of codes called error correcting codes. The basic principle is that instead of sending the exact message, Alice sends a modified longer version of the message so that even if some parts are corrupted Bob can recover what she meant. Fountain codes are based on principles of linear algebra, and were actually developed right here at Berkeley! The company that commercialized them, Digital Fountain, (started by a **Berkeley grad, Mike Luby**), was later acquired by Qualcomm. In this problem, we will explore some of the underlying principles that make fountain codes work in a very simplified setting.

The message that Alice wants to send to Bob are the three numbers a, b, and c. The message vector representing these numbers is $\vec{m} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$. Figure 2 shows how Alice's message is encoded in a transmission vector and how Bob's received vector may have some corrupted symbols.

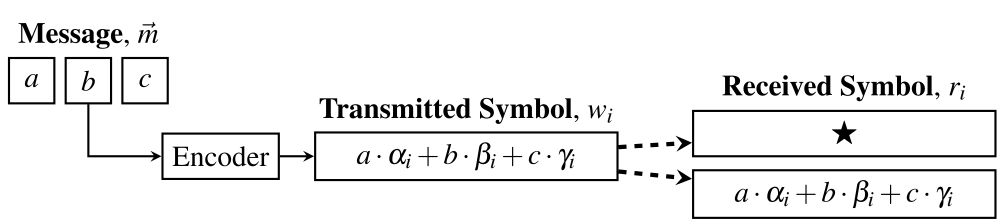


Figure 2: Each symbol in a transmission vector $\vec{w}$ is a linear combination of a, b, and c. Each transmitted symbol, w_i, is either received exactly as it was sent, or it is corrupted. A corrupted symbol is denoted by ★. The ith row of a symbol generating matrix G determines the values of α_i, β_i, and γ_i.

(a) Since Alice has three numbers she wishes to send, she could transmit six symbols in her transmission vector for redundancy. This transmission strategy is called the "repetition code".

If Alice uses the repetition code, her transmission vector is $\vec{w} = \begin{bmatrix} a \\ b \\ c \\ a \\ b \\ c \end{bmatrix}$.

As depicted in Figure 2, the received vector may have corrupted symbols. For example, suppose only the first symbol was corrupted, then Bob would receive the vector $\vec{r} = \begin{bmatrix} \bigstar \\ b \\ c \\ a \\ b \\ c \end{bmatrix}$, where the $\bigstar$ symbol represents a corrupted symbol.

Using the repetition code scheme, give an example of a received vector $\vec{r}$ with only two corrupted symbols such that a is unrecoverable but b and c are still recoverable.

Solution:

If Bob receives the vector $\vec{r} = \begin{bmatrix} \bigstar \\ b \\ c \\ \bigstar \\ b \\ c \end{bmatrix}$ then there is no way for him to recover a.

(b) Alice can generate $\vec{w}$ by multiplying her message $\vec{m}$ by a matrix. Write a matrix-vector multiplication that Alice can use to generate $\vec{w}$ according to the repetition code scheme. Specifically, find a "generating" matrix G_R such that $G_R\vec{m} = \vec{w}$, where $\vec{w} = \begin{bmatrix} a \\ b \\ c \\ a \\ b \\ c \end{bmatrix}$.

Solution:

Alice can use the follow symbol generating matrix G_R to generate $\vec{w}$:

$$G_R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(c) Instead of a repetition code, it is also possible to use other codes (e.g. fountain codes). Alice and Bob can choose any symbol generating matrix, as long as they agree upon it in advance. Each different matrix represents a different "code." Alice's TA recommends using the symbol generating matrix G_F:

$$G_F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Alice then uses the symbol generating matrix G_F to produce a new transmission vector: $G_F\vec{m} = \vec{w}$.

Suppose Bob receives the vector $\vec{r} = \begin{bmatrix} 7 \\ \bigstar \\ \bigstar \\ 3 \\ 4 \\ \bigstar \\ \bigstar \end{bmatrix}$, which is a corrupted version of $\vec{w}$.

Write a system of linear equations that Bob can use to recover the message vector $\vec{m}$. Solve it to recover the three numbers that Alice sent.

Hint: Consider the rows of G_F that correspond to the uncorrupted symbols in $\vec{r}$.

Solution:

In order to recover Alice's message $\vec{m}$, Bob needs to solve the following equation:

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \vec{m} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \\ 4 \end{bmatrix}$$

Bob can solve this system by writing the system in augmented matrix form and using Gaussian elimination:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 7 \\ 1 & 1 & 0 & 3 \\ 1 & 0 & 1 & 4 \end{array}\right]$$

$$\begin{array}{l} \text{Row 2: subtract Row 1} \\ \text{Row 3: subtract Row 1} \end{array} \implies \left[\begin{array}{ccc|c} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & -3 \end{array}\right]$$

So Bob recovers the message:

$$\vec{m} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 7 \\ -4 \\ -3 \end{bmatrix}$$

(d) We explore one example case where Alice and Bob agree to use G_F (i.e. $G_F\vec{m} = \vec{w}$) and there are three corruptions, so Bob receives four uncorrupted symbols.

Suppose Bob receives $\vec{r} = \begin{bmatrix} 1 \\ \bigstar \\ 3 \\ \bigstar \\ 4 \\ \bigstar \\ 9 \end{bmatrix}$. Can you determine the message $\vec{m}$ that Alice sent?

Solution: If Bob receives the odd symbols, this corresponds to the following system of equations:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \vec{m} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 4 \\ 9 \end{bmatrix}$$

This can be expressed as an augmented matrix and solved using Gaussian elimination:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 3 \\ 1 & 0 & 1 & 4 \\ 1 & 1 & 1 & 9 \end{array}\right]$$

$$\begin{array}{l} \text{Row 3: subtract Row 1} \\ \text{Row 4: subtract Row 1} \end{array} \implies \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 1 & 1 & 8 \end{array}\right]$$

$$\text{Swap Row 2 and Row 4} \implies \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 8 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$\begin{array}{l} \text{Row 2: subtract Row 3} \\ \text{Row 4: subtract Row 3} \end{array} \implies \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

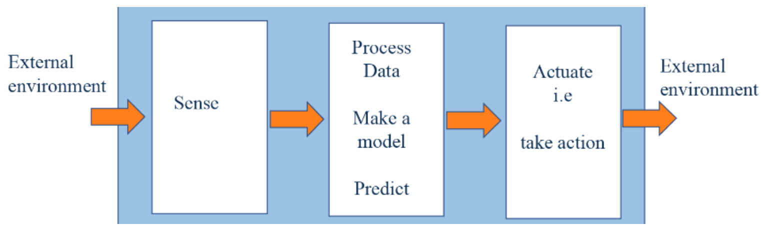
So Bob recovers the message:

$$\vec{m} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 3 \end{bmatrix}$$

Note: it can be shown that receiving *any* four uncorrupted symbols when Alice is using G_F is enough to recover Alice's message. On the other hand, we showed in part (a) that receiving *any* four uncorrupted symbols using G_R does not guarantee we can recover Alice's message. This is why, in practice, we would prefer to use the fountain code G_F instead of the repetition code G_R — it is a more reliable way to send messages.

6. **Group Problem: Explore an Interesting Application of EECS** This problem is meant to be solved in your study groups and start a discussion. The conversation you have is more important to us than the answers you write down to this problem. If you chose to not have a study group, you may do this on your own, or discuss with other students during homework party.

As a group, identify an exciting technology you would like to build. In this problem, you will go through the design process for this technology by identifying the required sensing, processing and actuation components. Some examples of technologies you could consider are the step counters on phones, GPS, self-driving cars, a robotic drummer accompanist, a fitbit, an advanced pacemaker, a hearing aid, touchscreens, an automated fruit picker that picks ripe fruit, augmented reality for emotional awareness, a car that can apologize to defuse road rage, a friend bot, a pollution cleaning-bot etc.



(a) What kind of information does your technology need to collect from the external environment? How is this collected? Are any sensors used? If so, what physical quantities do they measure?

(b) How do you propose to process the data obtained from the sensor measurements? What would you need to model? Would you have to make any predictions from the data? What techniques would you use?

(c) What actions would you want to take in the real world using your prediction or processed data? Describe the actuation mechanisms, if applicable, in this technology.

(d) What are the potential social implications of this technology? Good? Bad? Transformative? Redistributive?

Solution: Here is a sample response. Any sincere attempt at this problem should be given full credit for self-grades.

- One interesting application is step counters on phones. It is a challenging problem because you need to differentiate between actual walking motion vs. movements while the person is standing still. A key sensor used in this problem is an accelerometer, which measures accelerations experienced by the phone. You could also consider using other sensors, if available, like GPS to help with estimating motion when outdoors.
- From the raw acceleration time series, you can gather some relevant features like the magnitude of acceleration, time between peaks in the signal, etc. These can then be used to try to classify when a person is walking or standing still. For example, if the acceleration signal is roughly constant for a long time, then the person may be standing still or in a car.
- Although there is no traditional actuation mechanism in this technology, some possibilities are as follows: the graphical user interface (app), reminders to the user "to get up and walk", etc.
- Here is a cool reference paper on this topic:
  Kwapisz et al., Activity Recognition using Cell Phone Accelerometers, SensorKDD 2010.
  `https://dl.acm.org/doi/pdf/10.1145/1964897.1964918.`

- Step counters have had positive social impact by making many people more aware of their fitness levels and more motivated to exercise.

7. **Homework Process and Study Group**

   Who did you work with on this homework? List names and student ID's. (In case you met people at homework party or in office hours, you can also just describe the group.) How did you work on this homework? If you worked in your study group, explain what role each student played for the meetings this week.

   Solution:

   I first worked by myself for 2 hours, but got stuck on problem 5. Then I met with my study group

   XYZ played the role of facilitator ... etc. We were still stuck on problem 5 so we went to office hours to talk about the problem.

   Then I went to homework party for a few hours, where I finished the homework.