EECS 16A
Correlation and Classification
Admin

• Midterm 2 done! Yay!
• Redo due Monday – must complete to be eligible for clobber
  FRIDAY!!!
GPS 'trilateration': we can find our position in 2D world
- if we know our distance to 3 satellites
  + the satellite positions are known
  + the satellites are not collinear

Measuring distance to a satellite:
- The satellite is constantly putting out a 'song' (signal) played on repeat (one-way transmit)
- Our receiver has a synchronized clock and tries to figure out how delayed the signal is due to travel time from the satellite to receiver

\[ d = v \cdot t \]

For GPS, \( v = 3 \times 10^8 \text{ m/s} \)

Example:

![Diagram showing signal transmission and reception](image)

Signal from Satellite:

0 1 2 3 4 5 6 7 8 9 ...

Receives signal:

0 1 2 3 4 5 6 7 8 9 ...

Time delay = travel time

\( t = \text{samples} \)

Shift/delay is proportional to distance

\( t = \frac{d}{v} \)

Signal (song, gold code): \( \hat{s} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix} \)

Received signal: \( \hat{r} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \end{bmatrix} \)

All possible shifted vectors:

\( \hat{s}_0 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix} \)
\( \hat{s}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} \)
\( \hat{s}_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \end{bmatrix} \)

Since \( \hat{r} = \hat{s}_1 \), the delay was 2 samples.

Next, we wanted a robust way to find which of the shifted vectors is most similar to \( \hat{r} \).

What is \( k \) when \( \| \hat{r} - \hat{s}_k \| \) is minimum?

Norm of a vector \( \vec{x} \) is \( \| \vec{x} \| = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2} \)

Length, magnitude
Then, we calculated that $\| \mathbf{r} - \mathbf{S}_0 \|^2$ is minimum when $\mathbf{r}^T \mathbf{S}_0$ is maximum with respect to $\mathbf{S}_0$.

**Inner Product**

The dot product or inner product of vectors $\mathbf{x}$ and $\mathbf{y}$ is $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T \mathbf{y}$

$$= x_1 y_1 + x_2 y_2 + \ldots + x_n y_n$$

$$= \sum_{i=1}^{n} x_i y_i$$

It measures how aligned/similar two vectors are.

**Cauchy-Schwarz Inequality**

$$\langle \mathbf{x}, \mathbf{y} \rangle \leq \| \mathbf{x} \| \| \mathbf{y} \|$$

**Properties of inner products:**

1. $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle$
2. $\langle \mathbf{x} + \mathbf{z}, \mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{z}, \mathbf{y} \rangle$
3. $\langle \alpha \mathbf{x}, \mathbf{y} \rangle = \alpha \langle \mathbf{x}, \mathbf{y} \rangle$
4. $\langle \mathbf{x}, \mathbf{x} \rangle \geq 0$ (unless $\mathbf{x} = \mathbf{0}$ then $\langle \mathbf{x}, \mathbf{x} \rangle = 0$)

So, to find the shift, we need to compute inner products of the received signal with all possible shifts of the sent signal from the satellite. The values of these inner products arranged into a new signal is the cross-correlation:

**Cross-correlation** defined as $\text{Corr}_\mathbf{r}(\mathbf{s})[k] = \sum_{i=0}^{n} r(i) \cdot s(i-k)$ is the shift index!

$\mathbf{r} = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$

$\mathbf{s} = \begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{bmatrix}$

$\mathbf{r}^T \mathbf{s} = 2 \cdot 2 + 1 \cdot 3 + 1 \cdot 4 + 0 \cdot 5 + 1 \cdot 6 = 15$

$\mathbf{r}^T (\mathbf{s} - \mathbf{S}_0)$

$\| \mathbf{r} - \mathbf{S}_0 \|^2$ is minimized when $\mathbf{r}^T \mathbf{S}_0$ is maximized.

$\langle \mathbf{x}, \mathbf{y} \rangle$ tells you what error is.

$\| \mathbf{x} \| \| \mathbf{y} \|$ tells you how much.

$L$ is basically, each value in the correlation function is the inner product for a given shift value.

Largest value tells you what shift is!
Example:
\[
\text{corr}_x(\mathbf{s}) = \begin{bmatrix} 1 & 2 & 6 & 2 \end{bmatrix}
\]

Why is the inner product a good metric for which shift is correct?
- Noise robust (works even when dog barks)
- Can handle attenuation (if song is quieter than expected)
- Will work well even when multiple satellites 'on'!
  (receiver hears linear combo of multiple songs)

What signals make good songs?
- Want shifted versions to be uncorrelated
  i.e. ideal \( \text{corr}_x(\mathbf{s}) \) is orthogonal
  (almost)

Ex.
\[
\begin{align*}
\mathbf{s} & = \begin{bmatrix} 1 \end{bmatrix} \times \text{BAD} \\
\mathbf{s}' & = \begin{bmatrix} 1 & 2 \end{bmatrix} \times \text{BAD (repeat-5)} \\
\mathbf{s}'' & = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{bmatrix} \checkmark \text{good}
\end{align*}
\]
**Satellite classification:**

Which satellite am I talking to?

Example:

signature: \( S_A = \begin{bmatrix} -1 \\ -1 \\ \end{bmatrix} \)

"Gold code"

\[ \| S_A \| = \sqrt{5} \]

\( S_B = \begin{bmatrix} -1 \\ -1 \\ 0 \\ -1 \end{bmatrix} \)

\[ \| S_B \| = \sqrt{5} \]

\( 2 \) vectors in \( 3D \) space at some angle.

S: receive \( \hat{r} \) and want to know which satellite it's from:

\[ \hat{r} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \text{ - this is } S_A \text{ shifted by } 0 \text{ samples} \]

But in real life, there is noise, so we might receive something more like this:

\[ \hat{r} = \begin{bmatrix} 0.9 \\ 1.1 \\ -1.2 \\ 0 \end{bmatrix} = S + n \]

\( n \) may be \( S_A, S_B \) or a linear combo.

If just one satellite:

Want to know if \( \hat{r} \) is "closer" to \( S_A \) or \( S_B \)?

**Classify with noisy signal (classic problem)**

Which signal \( S_A \) or \( S_B \) is "closest" to \( \hat{r} \)?

\[ \hat{e}_A = \hat{r} - S_A \]

\[ \hat{e}_B = \hat{r} - S_B \]

Find satellite for which \( \hat{e} \) is minimized in terms of the norm:

Find which \( S_i \) minimizes \( \| \hat{e} \|_2 \)

\[ \| \hat{e} \|_2^2 = \langle \hat{e}, \hat{e} \rangle = \hat{e}^T \hat{e} \]

\[ = \langle \hat{r} - S_i, \hat{r} - S_i \rangle \]

\[ = \langle \hat{r}, \hat{r} \rangle - 2 \langle \hat{r}, S_i \rangle + \| S_i \|_2^2 \]

\[ = \| \hat{r} \|_2^2 + \| S_i \|_2^2 - 2 \langle \hat{r}, S_i \rangle \]

\[ \text{classify \( S_i \) by finding \( S_i \) that minimizes \( \| \hat{e} \|_2^2 \)} \]
So the problem amounts to minimizing $-2 \langle \hat{s}_n, \hat{r} \rangle$ over all $n$

Equivalent: maximize $2 \langle \hat{s}_n, \hat{r} \rangle$ over all $n$ (collinear $\rightarrow$ max $\langle , \rangle$)

Algorithm: For all satellites $\hat{s}_n$, compute $\langle \hat{r}, \hat{s}_n \rangle$. Find largest value.

Multiple satellites transmitting at once:

Each is sending its 'song' via EM waves obeying principle of superposition:

$$\hat{r} = \hat{s}_A + \hat{s}_B + \hat{n}$$

We can check inner product to see how similar received signal is to $\hat{s}_A$:

$$\langle \hat{r}, \hat{s}_A \rangle = \langle \hat{s}_A + \hat{s}_B + \hat{n}, \hat{s}_A \rangle = (\hat{s}_A + \hat{s}_B, \hat{n})^T \cdot \hat{s}_A$$

$$= \hat{s}_A^T \hat{s}_A + \hat{s}_B^T \hat{s}_A + \hat{n}^T \hat{s}_A$$

$$= \langle \hat{s}_A, \hat{s}_A \rangle + \langle \hat{s}_B, \hat{s}_A \rangle + \langle \hat{n}, \hat{s}_A \rangle$$

Now try $\langle \hat{r}, \hat{s}_C \rangle = \langle \hat{s}_A + \hat{s}_B + \hat{n}, \hat{s}_C \rangle$

$$= (\hat{s}_A, \hat{s}_C) + (\hat{s}_B, \hat{s}_C) + (\hat{n}, \hat{s}_C)$$

Can set a threshold for detecting $\hat{s}_A$:

if $\langle \hat{r}, \hat{s}_A \rangle \geq$ threshold, then $\hat{s}_A$ detected!

Does the shift matter?

What signals make good songs?

(- want shifts to be uncorrelated and two songs to be uncorrelated at all shifts)

Ex. $\hat{s}_A = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ $\hat{s}_B = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ X BAD! (one is shifted version of other)