\[
\begin{bmatrix}
\cos 90^\circ & \sin 90^\circ \\
-\sin 90^\circ & \cos 90^\circ \\
\end{bmatrix}
\begin{bmatrix}
a_1 \\
a_2 \\
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
\end{bmatrix}
\]
Admin

- First imaging lab with hardware!
- Keep on top of your homework
- If you’re lost, go to office hours
Summary: equivalent statements

• $Ax=b$ has a unique solution
• The columns of $A$ are linearly independent
• $A$ is invertible
• Every column in any row echelon form has a pivot
• Row reduced echelon form of $A$ leads to identity matrix
Proofs are hard! Here's some tips:

1) **Write out the statement** precisely in mathematical terms/equations
   - note the direction of implication (“if” →”then”)

2) Try a **simple example** to see if you can find a pattern
   - scribble, doodle, try weird things
   - write out related theorems, definitions

3) **Manipulate** both sides to see what goes in the “middle”
   - simplify complex notation
   - justify each step!

4) Know the **different styles** of proofs you can try:
   - constructive proofs
   - proofs by contradiction
Theorem: If the cols of $A$ are lin. dep., then $A\vec{x} = \vec{b}$ does not have a unique soln. 

Known: $A = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}$ n cols

Indep., so

\[ \vec{a}_k = c_1 \vec{a}_{1k} + c_2 \vec{a}_{2k} + \cdots + c_{k-1} \vec{a}_{(k-1)k} + c_{k+1} \vec{a}_{(k+1)k} + \cdots + c_n \vec{a}_{nn} \]

Wnt: $A\vec{x} = \vec{b}$ does not have a unique soln.

I.e. there is no soln. or there are 2

Assume opposite -> that there is soln $x_1 \ldots x_n$ (unique) -> show this is possible

\[ \text{Let's assume } \vec{x}_1 \text{ is unique soln then prove the opposite.} \]
\[ \text{Show: if } \vec{x}_1 \text{ is a solution, then there others exist:} \]
\[ \text{we know } A\vec{x}_1 = \vec{b} \]
\[ A\vec{x}_1 + \vec{w} = \vec{b} \]
\[ A(\vec{x}_1 + \vec{w}) = \vec{b} \]
\[ \text{So } \vec{x}_1 + \vec{w} \text{ is also a soln!} \]
\[ \checkmark \text{ yay!} \]

QED

Note: $A$ is given, so can't be zero
Let's think of matrix as an operator that transforms one vector into another vector.

\[ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix} \] 

Reflection matrix.

In particular, matrices are linear transformations. Will it work for any \((x, y)\)? Try it!

\[ \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix} \]

Linear Transformations: preserve addition & scalar multiplication.

\( f: \) is a linear transformation if
\[ f(\alpha x + \beta y) = \alpha f(x) + \beta f(y) \]
\[ f(\alpha \cdot x) = \alpha \cdot f(x) \quad \alpha \in \mathbb{R} \text{ scalar} \]

\( \cos \theta \) is linear
\( \alpha^2 \) is not linear.

Does matrix-vector multiplication satisfy lin. trans. defn.?

\( l(x) = Ax + \beta y \) if \( A \) is a matrix

\( l(x) = \alpha x \) if \( \beta = 0 \)

\( \begin{bmatrix} \cos 90^\circ & \sin 90^\circ \\ -\sin 90^\circ & \cos 90^\circ \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \)

Vectors can be used to represent the state of a system.

\( \begin{bmatrix} x(t) \\ y(t) \\ v(t) \end{bmatrix} \) represents position, velocity, etc.

What if it's changing in time?

\( \begin{bmatrix} x(t) \\ y(t) \\ v(t) \end{bmatrix} \) represents state at time \( t \).
New application/example system:
A system of water reservoirs & pumps:

- tanks of water
- what is the state of the system?

\[ \mathbf{x}(t) = \begin{bmatrix} x_A(t) \\ x_B(t) \\ x_C(t) \end{bmatrix} \]
\[ \mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \]

Say we connect these using pumps. Pumps run every time clock ticks (e.g. every second)

- Every time pump runs, all water from A moves back to A
- Every time pump runs all water from B \(\rightarrow\) C and all water C \(\rightarrow\) B

How can I represent this mathematically?

\[
\begin{align*}
x_A(t+1) &= x_A(t) \\
x_B(t+1) &= x_C(t) \\
x_C(t+1) &= x_B(t)
\end{align*}
\]

System of equations that describes evolution of the state over time:

Write in matrix form:

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
x_A(t) \\
x_B(t) \\
x_C(t)
\end{bmatrix} =
\begin{bmatrix}
x_A(t+1) \\
x_B(t+1) \\
x_C(t+1)
\end{bmatrix}
\]

\[ A \cdot \mathbf{x}(t) = \mathbf{x}(t+1) \]

State transition matrix.

What if I run pumps twice? Back to original! (\(A, B, C\) swap twice)
Conservative system means no water is lost or gained.
Graph Representation

Ex: Reservoirs and Pumps

I have 3 reservoirs: A, B, C and I want to keep track of how much water is in each

When I turn on some pumps, water moves between the reservoirs.

Where the water moves and what fraction is represented by arrows.

Nodes

Edges

“directed” graph because arrows have a direction

Need to label that too...

Can you tell me how much water in each after pumps start?

Need to know initial amounts
Say initial water levels (before pumps start) are $s = [s_1, s_2, s_3]$. How much water in reservoirs (nodes) after pumps run?

We are looking for solution $\tilde{s} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$:

- Water in reservoir 1 after pumps
- Water in reservoir 2 after pumps
- Water in reservoir 3 after pumps

Read off graph:

\[
\tilde{y} = \begin{bmatrix}
\frac{1}{2} s_1 + \frac{1}{3} s_2 + \frac{1}{6} s_3 \\
\frac{1}{2} s_1 + \frac{1}{3} s_2 + \frac{1}{3} s_3 \\
\frac{1}{2} s_1 + \frac{1}{3} s_2 + \frac{1}{3} s_3
\end{bmatrix} = S \begin{bmatrix} \frac{1}{2} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix} + S_2 \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix} + S_3 \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}
\]

- Column view → outflows
- Row view → inflows

Useful representation tool.

What else could nodes and edges represent?

E.g., people & flow; traffic
- Money & purchases
- Etc.

Does the total outflow have to equal 1?

No, but the total output must be conserved.

There is a sink (sink) e.g., $P_{\text{fat}}$ → maybe it's not conserved?

R-system is applied to solve:

\[
\tilde{y} = P \tilde{s}
\]

This is a matrix-vector multiplication.