$$\begin{bmatrix} \cos 90^\circ & \sin 90^\circ \\ -\sin 90^\circ & \cos 90^\circ \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \underline{a_2} & \underline{a_1} \end{bmatrix}$$

EE16A

Linear Transformations

Admin

- First imaging lab with hardware!
- Keep on top of your homework
- If you’re lost, go to office hours

Summary: equivalent statements

- Ax=b has a unique solution
- The columns of A are linearly independent
- A is invertible
- Every column in any row echelon form has a pivot
- Row reduced echelon form of A leads to identity matrix

Proofs are hard! Here's some tips:

1) **Write out the statement** precisely in mathematical terms/equations
   - note the direction of implication ("if" → "then")

2) Try a **simple example** to see if you can find a pattern

   - scribble, doodle, try weird things

   - write out related theorems, definitions

3) **Manipulate** both sides to see what goes in the "middle"

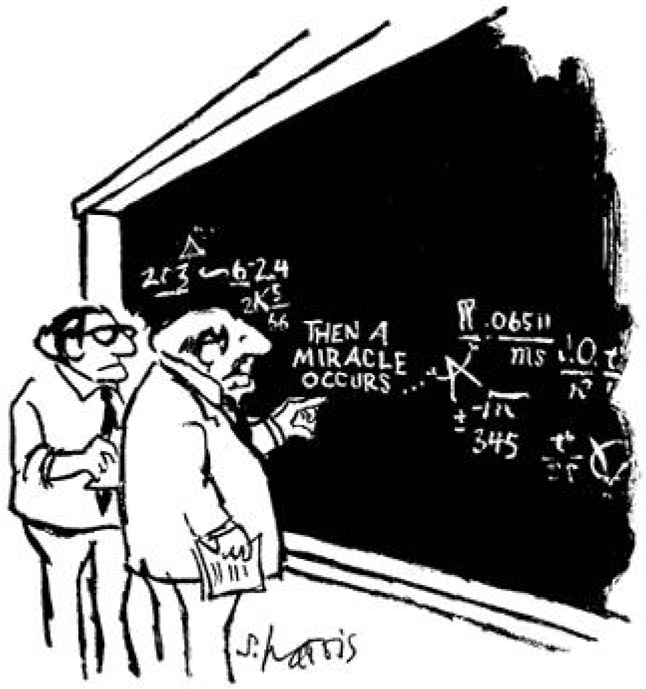
   - simplify complex notation

   - justify each step!

4) Know the **different styles** of proofs you can try:

   - constructive proofs

   - proofs by contradiction



"I think you should be more explicit here in step two."

Theorem: If the cols of A are lin. dep., then $A\vec{x}=\vec{b}$ does not have a unique sol'n. → no sol'n
→ or if 1 sol'n, then more

Known:

$$A = \begin{bmatrix} | & | & & | \\ \vec{a}_1 & \vec{a}_2 & \dots & \vec{a}_n \\ | & | & & | \end{bmatrix} \text{ n cols}$$

lindep., so

defn lin. dep

$$\vec{a}_k = c_1\vec{a}_1 + c_2\vec{a}_2 + \dots c_{k-1}\vec{a}_{k-1} + c_{k+1}\vec{a}_{k+1} \dots + c_n\vec{a}_n$$

Want: $A\vec{x}=\vec{b}$ does not have a unique sol'n.

i.e. there is no sol'n or there are 2+

Assume opposite → that there is sol'n x_* (uniqe) → show not possible

how to connect to $A\vec{x}=\vec{b}$?

write in terms of matrix A?

To write col $\vec{a}_k$ in terms of A:

$$\begin{bmatrix} | & | & & | \\ \vec{a}_1 & \vec{a}_2 & \dots & \vec{a}_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \leftarrow \text{pick out kth col} \Rightarrow \vec{a}_k$$

$$\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ 0 \\ \vdots \\ c_n \end{bmatrix} \leftarrow \text{delete kth col} \rightarrow \text{RHS of } *$$

$$A \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} = A \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ 0 \\ \vdots \\ c_n \end{bmatrix} \leftarrow \text{kth position}$$

or equiv.:

$$\text{kth position} \rightarrow \quad A \underbrace{\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ -1 \\ \vdots \\ c_n \end{bmatrix}}_{\text{call this } \vec{w}} = 0 \qquad A\vec{w} = 0$$

Now $A\vec{w}=\vec{0}$, but $\vec{w}$ is not zero vector b/c of -1, so no trivial sol'n.

Let's assume $\vec{x}_*$ is unique sol'n then prove the opposite.

Show: if $\vec{x}_*$ is a solution, then there others exist:

We know $A\vec{x}_* = \vec{b}$

$$A\vec{x}_* + \vec{0} = \vec{b}$$

shouldn't change it

$$A\vec{x}_* + A\vec{w} = \vec{b}$$

$$A(\vec{x}_* + \vec{w}) = \vec{b}$$

so $\vec{x}_* + \vec{w}$ is also a sol'n!

✓ yay!

QED

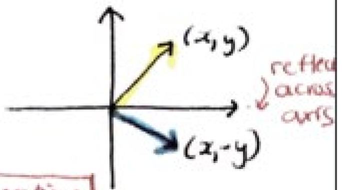
quod et demonstrandum

"that which was to be shown"

Note A is given so can't be zero

· Let's think of matrix as an operator that 'transforms' one vector into another vector.

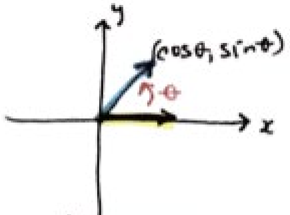
e.g. $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix}$ Reflection Matrix.



In particular, matrices are $\boxed{\text{linear transformations}}$

will it work for any (x, y)?? Try it!

e.g. $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$



$\boxed{\text{Linear Transformations}}$ → preserves addition & scalar multiplication

→ that's why it's called Linear Algebra

$\boxed{\begin{array}{l} f: \text{ is a linear transformation if} \\ f(\vec{x}+\vec{y}) = f(\vec{x}) + f(\vec{y}) \\ f(\alpha \cdot \vec{x}) = \alpha f(\vec{x}) \quad \alpha \in \mathbb{R} \text{ (scalar} \end{array}}$

e.g. $f(\vec{x}) = 2\vec{x}$ is linear

$f(\vec{x}) = x^2$ is not linear

Does matrix-vector multiplication satisfy lin. trans def'n?

↳ is $A(\vec{x}+\vec{y}) = A\vec{x} + A\vec{y}$ if A is a matrix ✓

and $A(\alpha\vec{x}) = \alpha(A\vec{x})$ ✓

Vectors can be used to represent the $\boxed{\text{state}}$ of a system

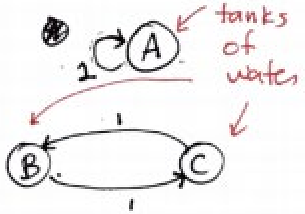
e.g. the "state" of a car: $\vec{s} = \begin{bmatrix} x \\ y \\ v \end{bmatrix}$ ← x position, ← y position, ← velocity } you could also use momentum, if interested, take 168, 127, 128

What if it's changing in time? $\vec{s}(t) = \begin{bmatrix} x(t) \\ y(t) \\ v(t) \end{bmatrix}$

$$\begin{bmatrix} \cos 90^\circ & \sin 90^\circ \\ -\sin 90^\circ & \cos 90^\circ \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \underline{\quad} & \underline{\quad} \end{bmatrix}$$

New application/example system:

A system of water resevoirs & pumps:

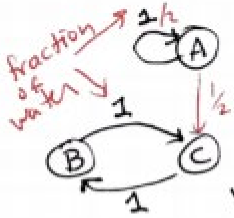


What is the state of the system?

$$\vec{x}(t) = \begin{bmatrix} x_A(t) \\ x_B(t) \\ x_C(t) \end{bmatrix}$$ ← water in A at time t

eg: $\vec{x}(1) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

Say we interconnect these using pumps. Pumps run every time clock ticks (e.g. every second)



Every time pump runs, all water from A moves back to A

Every time pump runs all water from B → C and all water C → B

How can i represent this mathematically?

$$\left.\begin{aligned} x_A(t+1) &= x_A(t) \\ x_B(t+1) &= x_C(t) \\ x_C(t+1) &= x_B(t) \end{aligned}\right\}$$ system of equations that describes ~~at~~ evolution of the state over time.

Write in matrix form:

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}}_{Q} \begin{bmatrix} x_A(t) \\ x_B(t) \\ x_C(t) \end{bmatrix} = \begin{bmatrix} x_A(t+1) \\ x_B(t+1) \\ x_C(t+1) \end{bmatrix}$$

$$\boxed{Q \cdot \vec{x}(t) = \vec{x}(t+1)}$$ Q is state transformation mtx.

What if i run pumps twice? back to original! (A never changes B,C swap twice)

Write out mathematically:

$$\vec{x}(t+2) = Q\,\underbrace{\vec{x}(t+1)}$$

$$= Q\left(Q\cdot\vec{x}(t)\right) = \underbrace{(Q\cdot Q)}\,\vec{x}(t)$$

this can be one matrix so mtx-vector mult.

What is $Q\cdot Q$? Q^2!

What do you expect it equals? Identity

Matrix-Matrix Multiplication:

$$Q\cdot Q = \begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}\begin{array}{c}\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}\\ \quad\vec{b}_1\ \vec{b}_2\ \vec{b}_3\end{array}$$

* treat 2nd matrix as a set of col vectors

$$= \begin{array}{c}\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\\ A\vec{b}_1\ A\vec{b}_2\ A\vec{b}_3\end{array}$$

✓ Identity doesn't change state!

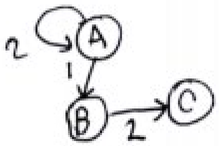
What if i run pumps $\underbrace{\text{3 times}}_{\text{same as once}}$? $\underbrace{\text{4 times}}_{\text{same as 2x}}$?

Note: Matrix multiply is not commutative (order matters), but in this case it is!

~~What~~ A billion times? No "steady state", just keeps flipping

If i take any mtx and square it, do i get identity? No!

Example:

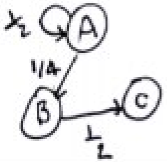


$$\begin{bmatrix}2&0&0\\1&0&0\\0&2&0\end{bmatrix}$$

"Non-conservative System"

→ generates water from nothing!

Example:



$$\begin{bmatrix}\frac{1}{2}&0&0\\\frac{1}{4}&0&0\\0&\frac{1}{2}&0\end{bmatrix}$$

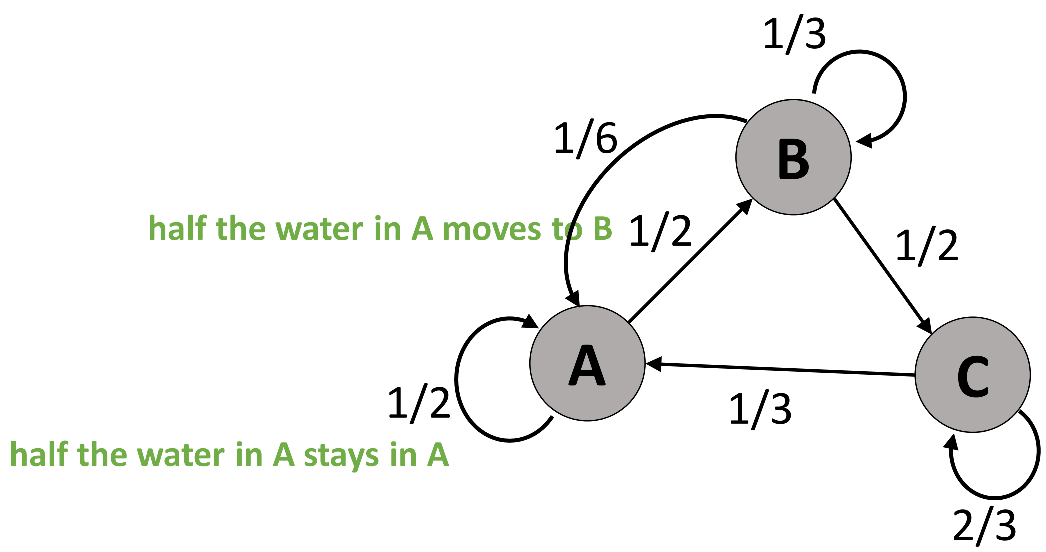
Also non-conservative

e.g. water evaporates.

Conservative system means no water is lost or gained.

Graph Representation

Ex: Reservoirs and Pumps



Nodes

I have 3 reservoirs: A,B,C
and I want to keep track of how much water is in each

When I turn on some pumps, water moves between the reservoirs.

Where the water moves and what fraction is represented by arrows.

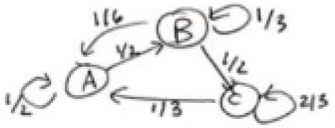
Edge weights **Edges**

"directed" graph because arrows have a direction

Where does the rest of the water in A go? Need to label that too…

Can you tell me how much water in each after pumps start? Need to know initial amounts

Reset Pumps Lec 02



Say initial water levels (before pumps start) are $\vec{s} = \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix}$ ← water in reservoir (A) ← water in reservoir (B)

$\boxed{\text{How much water in reservoirs (nodes) after pumps run?}}$

We are looking for solution $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ ← water in res. (A) <u>after pumps</u> ← water in res. (B) <u>after pumps</u>

Read off graph:

$$\vec{y} = \begin{bmatrix} \tfrac{1}{2}s_1 + \tfrac{1}{6}s_2 + \tfrac{1}{3}s_3 \\ \tfrac{1}{2}s_1 + \tfrac{1}{3}s_2 + 0s_3 \\ 0s_1 + \tfrac{1}{2}s_2 + \tfrac{2}{3}s_3 \end{bmatrix} = s_1\begin{bmatrix} 1/2 \\ 1/2 \\ 0 \end{bmatrix} + s_2\begin{bmatrix} 1/6 \\ 1/3 \\ 1/2 \end{bmatrix} + s_3\begin{bmatrix} 1/3 \\ 0 \\ 2/3 \end{bmatrix}$$

s_1 coeffs are <u>OUTFLOW</u> for Res (A) s_2 coeffs s_3 coeffs.

column view → OUTFLOWS

$$= \begin{bmatrix} 1/2 & 1/6 & 1/3 \\ 1/2 & 1/3 & 0 \\ 0 & 1/2 & 2/3 \end{bmatrix}\begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix}$$

<u>inflow</u> for Res (A)!

Row view → inflows

$$\vec{y} = \underbrace{P}_{\text{matrix}} \vec{s}$$

← system is applied to state

This is a matrix-vector multiplication!

useful representation tool.

$\boxed{\text{What else could \underline{nodes} and \underline{edges} represent?}}$

<u>e.g.</u> - people & flow/traffic
- money & purchases
- etc.

$$P_{matrix} = \begin{bmatrix} P_{A\to A} & P_{B\to A} & P_{C\to A} \\ P_{A\to B} & P_{B\to B} & P_{C\to B} \\ P_{A\to C} & P_{B\to C} & P_{C\to C} \end{bmatrix}$$

$\boxed{\text{Does the total outflow have to = 1?}}$

No, but then water not conserved! if cols total $\neq 1$

There is a <u>sink</u> (leak)   e.g. $P = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ → maybe it evaporated?