EE16A
Inversion
Invertibility brings justice!

Images released by Interpol in 2007 show the ‘unswirling’ of the internet pictures that led to the capture of Christopher Paul Neil.
I have 3 reservoirs: A, B, C and I want to keep track of how much water is in each. When I turn on some pumps, water moves between the reservoirs. Where the water moves and what fraction is represented by arrows.

Nodes

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Nodes

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Nodes

Where the water moves and what fraction is represented by arrows.

Edge weights

Note, all this stuff happens at one single time step!

“directed” graph because arrows have a direction.
Recap Last Class:

Pumps - example:

And I run pumps once, so \( t=0 \rightarrow t=1 \)

So columns represent outflows, rows represent inflows.

Does outflow have to be equal to inflow? to 1?
No, but then water is not conserved! (there is a source or sink)

What happens if I swap direction of arrows?
P gets transposed, NOT the inverse.

Recall, we can think of \( P \) matrix as a transformation of the state.

What if I run pumps twice?

Take output of first run and use as input for second run.

Example: \( \dot{x}(t=0) = \begin{bmatrix} 2 \\ 6 \end{bmatrix} \)

First run:

\[ \dot{x}(t=1) = P \cdot \dot{x}(t=0) = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \]

Second run:

\[ \dot{x}(t=2) = P \cdot \dot{x}(t=1) = P \cdot P \cdot \dot{x}(t=0) = P \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2.5 \\ 3.5 \end{bmatrix} \]

Is water conserved? Yes since sum of \( P \) col values = 1 for all cols.

What if I run pumps a bajillion times? Is there an equilibrium state, "steady state"?

I.e. does the water levels settle to a steady value?

Such that in steady state, \( \dot{x} \rightarrow \dot{x}_w \), we have \( \dot{x}_w = P \cdot \dot{x}_w \).
Written in matrix-vector multiplication form:

\[ \tilde{\mathbf{x}}^* = P \tilde{\mathbf{x}}^* \]

If it exists, then equilib. input = output

Rearrange: \[ P \tilde{\mathbf{x}}^* - \tilde{\mathbf{x}}^* = 0 \]

\[ P \tilde{\mathbf{x}}^* - I \tilde{\mathbf{x}}^* = 0 \]

\[ P \tilde{\mathbf{x}}^* = \tilde{\mathbf{x}}^* \]

↑ doesn't change anything cause \( I \tilde{\mathbf{x}}^* = \tilde{\mathbf{x}}^* \)

but matches up dimensions

\[ (P - I) \tilde{\mathbf{x}}^* = 0 \]

\[ \text{A form!} \]

\[ \begin{bmatrix} 1/2 & 1/6 & 1/3 \\ 1/2 & 1/3 & 0 \\ 0 & 1/2 & 2/3 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_{1^*}^* \\ x_{2^*}^* \\ x_{3^*}^* \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \]

\[ P \text{ From example} \]

Augmented Matrix form:

\[ \begin{bmatrix} -1/2 & 1/6 & 1/3 & 0 \\ 1/2 & -2/3 & 0 & 0 \\ 0 & 1/2 & -1/3 & 0 \end{bmatrix} \]

do G.E.

system has infinite sol'ns

\[ \tilde{x}^* = \begin{bmatrix} 8 \alpha \\ 6 \alpha \\ 9 \alpha \end{bmatrix} \]

for any scalar \( \alpha \in \mathbb{R} \)

Let's pick \( \alpha = 1 \)

\[ \tilde{x}^* = \begin{bmatrix} 8 \\ 6 \\ 9 \end{bmatrix} \]

steady state Solution

How can we check if it's correct?!

Plug into:

\[ P \cdot \tilde{x}^* = \begin{bmatrix} 1/2 & 1/6 & 1/3 \\ 1/2 & 1/3 & 0 \\ 0 & 1/2 & 2/3 \end{bmatrix} \begin{bmatrix} 8 \\ 6 \\ 9 \end{bmatrix} = \begin{bmatrix} 4 + 1 + 3 \\ 4 + 2 + 0 \\ 0 + 3 + 6 \end{bmatrix} = \begin{bmatrix} 8 \\ 6 \\ 9 \end{bmatrix} \]

output is same as input! ✓

Recap:

to find \( \tilde{x} \) at later time, apply \( P \) successively.

E.g. \( \tilde{x}(t+1) = P \cdot \tilde{x}(t) \)

\( \tilde{x}(t+2) = P^2 \cdot \tilde{x}(t) \)

\( \tilde{x}(t+\infty) = P^\infty \cdot \tilde{x}(t) \)?
What if I want to know the water levels at a previous time?

What is \( \hat{x}(t=-1) \)? Given I know \( \hat{x}(t=0) \),

\[ \hat{x}(t-1) \]

can write as \( \hat{x}(t-1) \)

The linear transformation that describes this is called the **Inverse**

denoted \( A^{-1} \) or \( P^{-1} \)

\[ \begin{align*} P^{-1} \hat{x}(t) &= P^{-1} \hat{x}(t-1) \\
&= \hat{x}(t-1) \end{align*} \]

The inverse of \( P \) 'undoes' what \( P \) did

Is it same as turning arrows backward? No! see discussion sec.

**Examples:**

What is the inverse of \( f(x) = 2x \)?

\[ g(x) = \frac{1}{2} x, \]

so \( f(g(x)) = x \)

Is \( f(x) = 0 \) invertible? No

Is eating a sandwich invertible? No (not really)

Is a scribble with iPad stylus invertible? Yes, with 'undo'

Basically, invertible means we can 'undo' function & recover input.

(think about tomography problem application)

**Example:**

System \( R: \)

\[ \begin{align*} \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} x_1 \cr x_2 \end{bmatrix} & = \begin{bmatrix} 0.5 & 1 \\
0 & 0.5 \end{bmatrix} \begin{bmatrix} x(t) \cr 0 \end{bmatrix} \\
= R \cdot \hat{x}(t) \end{align*} \]

System \( Q: \)

\[ \begin{align*} \begin{bmatrix} A^2 & B^2 \end{bmatrix} \begin{bmatrix} x_1 \cr x_2 \end{bmatrix} & = \begin{bmatrix} 0 & 2 \\
0 & 0 \end{bmatrix} \begin{bmatrix} x(t+1) \cr x(t+2) \end{bmatrix} \\
= Q \cdot \hat{x}(t) \end{align*} \]

Let's compute:

\[ \begin{align*} \hat{x}(t+1) &= R \cdot \hat{x}(t) \rightarrow \text{run } R \text{ system} \\
\hat{x}(t+2) &= Q \cdot \hat{x}(t+1) \rightarrow \text{then } Q \\
= Q \cdot R \hat{x}(t) \\
&= \begin{bmatrix} 2 & 0 \\
0 & 2 \end{bmatrix} \begin{bmatrix} 0.5 & 0 \\
0 & 0.5 \end{bmatrix} \hat{x}(t) \\
= \begin{bmatrix} 1 & 0 \\
0 & 1 \end{bmatrix} \hat{x}(t) \\
\end{align*} \]

\[ \text{nothing changes!} \]

Back to input...

**So** \( Q \cdot R = R \cdot Q = I \)

and \( Q \& R \) are inverses of each other
Definition of Inverse matrix: Let \( P, Q \) be square matrices

\( P \) is the inverse of \( Q \), and vice versa, if \( P \cdot Q = Q \cdot P = I \)

\( \) We say \( P = Q^{-1} \)

 mtx mult is generally \underline{not} commutative, but Inverses are.

Properties:

- Inverse is unique
- Any inverse is inverse on both left and right
- Inverse exist implies one unique solution to system

Example:

\[
\begin{align*}
\begin{pmatrix}
\frac{1}{2} \\
1 \\
\frac{1}{2}
\end{pmatrix}
& \xrightarrow{A} \begin{pmatrix} 1 & 0 \end{pmatrix} \\
& \xrightarrow{B} \begin{pmatrix} 1 & 0 \end{pmatrix}
\end{align*}
\]

What is \( P \) matrix?

Write out equations:

\[
\begin{align*}
X_A(t+1) &= X_B(t) \\
X_B(t+1) &= \frac{1}{2}X_A(t) + X_C(t) \\
X_C(t+1) &= \frac{1}{2}X_A(t)
\end{align*}
\]

\[
P = \begin{bmatrix} 0 & 1 & 0 \\
\frac{1}{2} & 0 & 1 \\
\frac{1}{2} & 0 & 0 \end{bmatrix}
\]

How to find inverse?

Want \( P^{-1} \) such that

\[
\begin{align*}
\vec{x}(t) &= P^{-1} \vec{x}(t+1) \\
\vec{x}(t+1) &= P \vec{x}(t)
\end{align*}
\]

\[
P \cdot P^{-1} = I
\]

But it's mtx-mtx mult., not mtx-vector like \( \vec{x} = \vec{b} \)?

treat each col as separate mtx-vector problem!

Could solve 3 G.E. \( Ax = b \) style problems for \( \hat{p}_1, \hat{p}_2, \hat{p}_3 \) then put into matrix

BUT steps of G.E. only depend on \( P \) so can do all at once! yay!
Theorem: If the cols of mtx A are lin. dep.,
then A is not invertible \((A^{-1} \text{ doesn't exist})\)

\[
P \implies q \implies \neg q \implies \neg P
\]

If A invertible \(\implies\) then cols of A are lin. indep.

Proof:

Known/Beginning

\[
A = \begin{bmatrix}
1 & 1 & 1 \\
\tilde{a}_1 & \tilde{a}_2 & \tilde{a}_3 \\
1 & 1 & 1
\end{bmatrix}
\]

cols lin. dep.

so can write by defn of lin. dep.
there exists
\[c_1 \tilde{a}_1 + c_2 \tilde{a}_2 + \cdots + c_n \tilde{a}_n = \mathbf{0}\]

and not all \(c_i\)'s are zero

need to connect cols to \(A^{-1}\)?

If we let \(A^{-1}\) exist:

\[A^{-1} \cdot A = AA^{-1} = I\]

can we write col or mtx. in terms of other?

Write col stuff as mtx:

\[
\begin{bmatrix}
\tilde{a}_1 & \tilde{a}_2 & \cdots & \tilde{a}_n
\end{bmatrix}
\begin{bmatrix}
c_1 \\
c_2 \\
\vdots \\
c_n
\end{bmatrix} = \mathbf{0}
\]

\[A \begin{bmatrix}
c_1 \\
c_2 \\
\vdots \\
c_n
\end{bmatrix} = \mathbf{0}\]

Now what? mult by \(A^{-1}\) on both sides on LEFT does not commute.

\[A^{-1}(A \begin{bmatrix}
c_1 \\
c_2 \\
\vdots \\
c_n
\end{bmatrix}) = A^{-1}\mathbf{0} \neq \mathbf{0}\]

\[\text{QED}\]
Finding a matrix inverse by **Gauss-Jordan Method**

- Similar process as Gauss Elim.
- Work to get into **reduced Row Echelon form (RREF)**

If \( A \) is invertible, want to find \( B = A^{-1} \) such that \( A \cdot B = I \)

Augmented matrix form:

\[
\begin{bmatrix}
A & I \\
\end{bmatrix} \xrightarrow{G.E.} \begin{bmatrix}
I & A^{-1} \\
\end{bmatrix}
\]

At end of G.E.

What if G.E. doesn't work?
- Then there is no inverse (or you made a mistake!)

Ex:

\[
\begin{bmatrix}
a & b & 1 & 0 \\
c & d & 0 & 1 \\
\end{bmatrix}
\]

Assume \( a \) is positive here.

\[
\begin{bmatrix}
a & b & 1 & 0 \\
0 & ad-bc & -c & a \\
\end{bmatrix}
\]

Row 2 - \( c \) times Row 1

Can't be zero or inverse doesn't exist

\[
\begin{bmatrix}
a & b & 1 & 0 \\
0 & 1 & \frac{-bc}{ad-bc} & \frac{a}{ad-bc} \\
\end{bmatrix}
\]

Row 1 \( \rightarrow \) Row 1 - \( b \) times Row 2

\[
\begin{bmatrix}
1 & 0 & \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\
0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \\
\end{bmatrix}
\]

\( R1 \) \( \rightarrow \) \( \frac{1}{a} \)

\[
\begin{bmatrix}
1 & 0 & \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\
0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \\
\end{bmatrix}
\]

Pull out scalar

\[
A^{-1} = \frac{1}{ad-bc} \begin{bmatrix}
d & -b \\
-c & a \\
\end{bmatrix}
\]

Formula for \( 2 \times 2 \) inverse