...AND THAT'S THE BEST FORMULA TO INCREASE YOUR PAGE RANKING ON GOOGLE. ANY QUESTIONS?
Recall: Equivalent Statements:

- Matrix A is invertible
- Ax=b has a unique solution
- A has linearly independent columns (A is full rank)
- A has a trivial nullspace
- The determinant of A is not zero
Jargon Roundup

• **range/span** of matrix $A$ is the set of all possible linear combinations of the column vectors (all the outputs it can get to)
Jargon Roundup

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• **rank** is the dimension of the span of the columns of matrix $A$

$$\dim(\text{colspan}(A)) = \# \text{ ind. cols.} = \# \text{ of cols. \_\_ \_ pivots} = \dim(\text{rowspan}(A))$$

\# lin. ind. cols $A = \# \text{ lin. indep. rows } A$
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• **nullspace** of matrix $A$ is the set of solutions to $Ax = 0$ (all the places it can’t get to).

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

**Defini:**

$$N(A) = \left\{ \vec{x} \mid A\vec{x} = \vec{0}, \vec{x} \in \mathbb{R}^n \right\}$$

**Rank-Nullity Theorem:**

$$\text{rank}(A) + \text{dim}(\text{null}(A)) = \# \text{ of cols of } A$$

$$\text{rank}(A) = 2$$

$$\text{dim}(\text{null}(A)) = 1$$

$$\mathbb{R}^3 \quad 3 \text{ dim total} \quad 2 + 1 = 3$$
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- **vector space** is a set of vectors connected by two operators ($+, \times$) that obeys the 10 axioms

A vector space $\mathcal{V}$ is a set of vectors and two operators that satisfy the following properties:

*Vector Addition*
- Associative: $\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$ for any $\vec{v}, \vec{u}, \vec{w} \in \mathcal{V}$.
- Commutative: $\vec{u} + \vec{v} = \vec{v} + \vec{u}$ for any $\vec{v}, \vec{u} \in \mathcal{V}$.
- Additive Identity: There exists an additive identity $\vec{0} \in \mathcal{V}$ such that $\vec{v} + \vec{0} = \vec{v}$ for any $\vec{v} \in \mathcal{V}$.
- Additive Inverse: For any $\vec{v} \in \mathcal{V}$, there exists $-\vec{v} \in \mathcal{V}$ such that $\vec{v} + (-\vec{v}) = \vec{0}$. We call $-\vec{v}$ the additive inverse of $\vec{v}$.
- Closure under vector addition: For any two vectors $\vec{v}, \vec{u} \in \mathcal{V}$, their sum $\vec{v} + \vec{u}$ must also be in $\mathcal{V}$.

*Scalar Multiplication*
- Associative: $\alpha(\beta \vec{v}) = (\alpha \beta) \vec{v}$ for any $\vec{v} \in \mathcal{V}, \alpha, \beta \in \mathbb{R}$.
- Multiplicative Identity: There exists $1 \in \mathbb{R}$ where $1 \cdot \vec{v} = \vec{v}$ for any $\vec{v} \in \mathcal{V}$. We call $1$ the multiplicative identity.
- Distributive in vector addition: $\alpha(\vec{u} + \vec{v}) = \alpha \vec{u} + \alpha \vec{v}$ for any $\alpha \in \mathbb{R}$ and $\vec{u}, \vec{v} \in \mathcal{V}$.
- Distributive in scalar addition: $(\alpha + \beta) \vec{v} = \alpha \vec{v} + \beta \vec{v}$ for any $\alpha, \beta \in \mathbb{R}$ and $\vec{v} \in \mathcal{V}$.
- Closure under scalar multiplication: For any vector $\vec{v} \in \mathcal{V}$ and scalar $\alpha \in \mathbb{R}$, the product $\alpha \vec{v}$ must also be in $\mathcal{V}$. 

Note 7
Jargon Roundup

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- **vector subspace** is a subset of vectors from a vector space that obey 3 properties

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**Definition 8.1 (Subspace):** A subspace $U$ consists of a subset of the vector space $V$ that satisfies the following three properties:

- Contains the zero vector: $\vec{0} \in U$.
- Closed under vector addition: For any two vectors $\vec{v}_1, \vec{v}_2 \in U$, their sum $\vec{v}_1 + \vec{v}_2$ must also be in $U$.
- Closed under scalar multiplication: For any vector $\vec{v} \in U$ and scalar $\alpha \in \mathbb{R}$, the product $\alpha \vec{v}$ must also be in $U$.  

**Note 8**
Jargon Roundup

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• **column space** is the span(range) of the columns of a matrix

• **row space** is the span of the rows of a matrix
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- **dimension** of a vector space is the number of basis vectors (degrees-of-freedom)
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- A **basis** for a vector space is a minimum set of vectors needed to represent all vectors in the space
Jargon Roundup

Most "efficient representation" of a vector space.

\[
\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0.5 \\ 0.7 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \pi \end{bmatrix} \} \Rightarrow \text{vector space } \mathbb{R}^2
\]

Basis set: \{ \begin{bmatrix} 0.5 \\ 0.7 \end{bmatrix}, \begin{bmatrix} \pi \\ 0 \end{bmatrix} \}

Def'n: Given a vector set \( \{ \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \} \)

\( \Rightarrow \) a set of vectors \( \{ \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \} \) is a BASIS SET

if

1. \( \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \) are lin. indep.
2. \( \forall \mathbf{v} \in \mathbf{V} \) (any vect) \( \Rightarrow \mathbf{v} = a_1 \mathbf{v}_1 + a_2 \mathbf{v}_2 + \ldots + a_n \mathbf{v}_n \)

A basis for a vector space is a minimum set of vectors needed to represent all vectors in the space
Today’s Jargon: determinant, eigenvalue, eigenvector

• We will use Google’s PageRank algorithm as a new application example to learn about these things!
\[ \text{Determinant} \quad \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc \]

\[ \text{det}(A) = |A| \]

\[ \det(A) \neq 0 \text{ when } A \text{ invertible} \]

\[ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{Area} = 0 \]

\[ \begin{pmatrix} 1 & -1 \\ 2 & 2 \end{pmatrix} \quad \not \text{invertible} \quad \det(A) = 0 \]

\[ A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \]

Compute area of parallelogram as:

\[ \begin{array}{c}
\text{Area} = (c+d)(a+b) - 2bc - ac - bd \\
\hline
\text{Area} = \frac{1}{2} ad - bc \quad \text{yay!}
\end{array} \]
Page rank

\[ \vec{x} = \begin{bmatrix} x_{\text{Stanford}} \\ x_{\text{UCB}} \end{bmatrix} \]

\[ Q = \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & 1 \end{bmatrix} \]

State transition matrix

Start with everyone at Stanford \( \vec{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \)

After one timestep: \( \vec{x}(1) = Q \cdot \vec{x}(0) \)

\[ = \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \]

After two timesteps: \( \vec{x}(2) = Q \cdot \vec{x}(1) \)

\[ = \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{4} \\ \frac{3}{4} \end{bmatrix} \]

After 3 timesteps: \( \vec{x}(3) = Q \cdot \vec{x}(2) \)

\[ = \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{4} \\ \frac{3}{4} \end{bmatrix} = \begin{bmatrix} \frac{1}{8} \\ \frac{7}{8} \end{bmatrix} \]

\( \vec{x}(t) = \begin{bmatrix} \left(\frac{1}{2}\right)^t \\ 1 - \left(\frac{1}{2}\right)^t \end{bmatrix} \)

\( \vec{x}(\infty) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \)

Everyone's at UCB!
What if instead \( \ddot{x}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \)?

\[ \ddot{x}(1) = Q \cdot \ddot{x}(0) = \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \]

Steady State Sol'n

In general,

\( \ddot{x}_{steady} = Q \cdot \ddot{x}_{steady} \)

\[ I \ddot{x}_{ss} = Q \cdot \ddot{x}_{ss} \]

\[ Q \cdot \ddot{x}_{ss} - I \ddot{x}_{ss} = \ddot{0} \]

\[ (Q - I) \ddot{x}_{ss} = \ddot{0} \]

Want to find \( \text{Null}(Q - I) \rightarrow \text{steady state sol'n.s.} \)

\[ \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & 0 \\ \frac{1}{2} & 0 \end{bmatrix} \]

\( \text{Null}(Q - I) : \begin{bmatrix} -\frac{1}{2} & 0 \\ \frac{1}{2} & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \)

\( \text{Null}(Q - I) = \begin{bmatrix} 0 \\ x_2 \end{bmatrix} = \text{span} \{ [0] \} \)

Eigen space of \( Q \) corresponding to eigenvalue \( 1 \rightarrow \text{steady state} \)
**Definition:** Let $Q$ be a square matrix, $\lambda \in \mathbb{R}$, if $x \neq 0$ such that $Q \cdot x = \lambda \cdot x$, then $\lambda$ is an eigenvalue of $Q$ and $\text{Null}(Q - \lambda I)$ is its eigenspace corresponding to $\lambda$.

Find eigenvalues and eigenspaces:

Find $x$ such that $Q \cdot x = \lambda \cdot x$.

Let $Q = \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & 1 \end{bmatrix}$

$Q \cdot x = \lambda \cdot I \cdot x$

$(Q - \lambda I) \cdot x = 0$

If we have non-trivial nullspace:

$\det(Q - \lambda I) = 0$

$(\frac{1}{2} - \lambda)(1 - \lambda) = 0$

$\lambda_1 = \frac{1}{2}$, $\lambda_2 = 1$

Characteristic polynomial