18.1 Introduction: Op-amps in Negative Feedback

In the last note, we studied how to use an op-amp as a comparator. We did this by assuming the internal gain of an op-amp, $A$, is very large (approaching infinity) and modelling the output as a piecewise function between the two supply rails $V_{DD}$ and $V_{SS}$. In reality though, this gain, $A$, is finite, so one may consider using this as an amplifier to amplify the input difference $(V_+ - V_-)$. However, since the gain, $A$, is very large (between 1000-100000 depending various factors), in practice, any input $V_+$ and $V_-$ with a slight difference between them will result in an output $V_{out} = A(V_+ - V_-)$ that is saturated or railed to either $V_{DD}$ or $V_{SS}$.

In this note, we will look at how to use op-amps to scale the input difference $(V_+ - V_-)$ without causing the op-amp to rail. We will first motivate this with a design example. Then, we will show techniques using negative feedback to build blocks that depend only on ratio of physical quantities as long as the op-amp has high enough (but not precise) internal gain.

18.2 Design example – Digital to Analog Converter (DAC)

A digital to analog converter (DAC) is a component that translates digital input signals into output analog voltages. Like an op-amp, DACs have supply rails $V_{DD}$ and $V_{SS}$, and the output voltage will be restricted between those values, i.e the maximum output voltage is $V_{DD}$ and the minimum is $V_{SS}$. Commonly, we model DACs as simply a voltage source with a series resistor. The symbol is shown below (left), along with an example model circuit (right).

Consider a song stored on your computer, encoded in some digital values (some sequence of 1s and 0s). We would like to convert it to some analog (continuous value) signal to play on a speaker. Below is a simple depiction of our speaker, which takes an input signal between 0V and 10V (as indicated by the arrow) and outputs some corresponding sound. We can model our speaker simply as a small resistor, 8Ω for example.
To design an audio system, the simplest thing we can do is directly connect the DAC to the speaker. The DAC takes in digital bits and converts them into an analog signal, then this signal is fed into a speaker.

Using the models from above, we can draw this complete circuit model:

We want the input voltage to the speaker to be between 0V and 10V (in order to make the speaker loud enough). To reach that maximum and minimum voltage, we may try to make $V_{DD} = 10$V and $V_{SS} = 0$V for the DAC. Unfortunately, these are not always available to us. Depending on the system, we may be restricted to using only 5V, 3.3V, or even less.

For this example, we will use $V_{DD} = 3.3$V and $V_{SS} = 0$V, realistic values for many computer systems. This means the maximum voltage the DAC can produce is 3.3V, and the minimum voltage the DAC can produce is 0V. From the circuit model above, we see that:

$$V_{speaker} = \frac{8}{8+1000} \times V_{TH}$$

which is much smaller than $V_{TH}$. However, to match the speaker’s input voltage range, we actually want to be able to map voltages from 0 to 3.3V at the DAC output to voltages from 0 to 10V at the speaker input, so $V_{speaker}$ should be three times (10/3.3) larger than $V_{TH}$ to scale the voltage correctly. This circuit isn’t going to do what we would like!

Instead of directly connecting the DAC and speaker, we need some intermediate circuit. This circuit needs to accomplish three things: it must provide a gain of roughly 3, it must not affect the output voltage of the
DAC (so that it does not change the signal), and it must be able to supply any current required by the speaker (so that the speaker does not affect the expected output voltage). So we want something like the following

This looks just like the internal of an op-amp! Now notice that based on what we know so far about op-amps as comparators, we can’t just scale the voltage linearly as we would have wished: if we connect an ideal op-amp (infinite internal gain) with $V_{DD} = 10V$ and $V_{SS} = 0V$, the output voltage would either be 10V or 0V, but not something in between. To achieve intermediate values and use the op-amp as an amplifier, we need another tool, which we will introduce in the next section.

18.3 Negative feedback

Negative feedback is used just about everywhere, including electronics, biology, mechanics, robotics, and more. Many systems and processes will drift into unknown or undesirable states if left unattended or uncontrolled. Negative feedback occurs when some function of the output of a system is fed back into the input, in order to correct the system and keep the output at some controlled value. Let’s turn this high level description into a more mathematical one.

Concretely, we want to get a certain known gain out of our op-amp. Currently we have an op-amp with some very large uncertain internal gain. We can describe this problem using a block diagram; a collection of drawings (mathematical in nature) that operate on quantities of interest using simplified representations.

Let’s take a look at a generic block diagram for negative feedback systems.

In this diagram, the arrows represent the direction of signals. If an arrow is going into a block, it is an input to that block. If the arrow points away from a block, it is an output of that block. The blocks labelled $A$ and $f$ are multiplier elements, i.e. they take some input $x$ and output that input multiplied by some factor $Ax$. The circle with a $+$ symbol is an addition element, i.e. it takes some inputs $a$ and $b$ and outputs the sum.
$a + b$. With addition elements, we may add signs to the inputs to signal if we want to add or subtract them. For example, in this diagram, the $-$ symbol next to the feedback signal signifies we will be subtracting it, i.e. error = input $-$ feedback.

In words, this diagram takes the difference between the input and the feedback, which is a scaled version of the output (i.e. $f$ times the output). This difference is called the error, and we apply gain $A$ on it to again produce the output.

The "negative" aspect of this feedback comes from taking the difference between the input and the feedback signal. Observe that if the feedback signal for some system is increased, the error (input - feedback) signal will decrease (move down), which then causes the output to go down as well, i.e., the loop has the ability to suppress the original change in the feedback signal!

Now we can get an intuitive idea of how negative feedback can be useful. When we want to get a system to have a desired output, negative feedback loops can help re-adjust to the value of the desired output when the output is too high or too low relative to the target value.

Now let’s see how negative feedback loops can be realized in op-amps. Consider the following circuit:

![Op-amp circuit diagram]

We want to map the components here to the feedback block diagram above. We have an input signal $V_{in}$ and an output $V_{out}$. The voltage divider formed by $R_1$ and $R_2$ multiplies our output $V_{out}$ to give us the feedback signal $V_{fb}$, which is connected to the $-$ terminal of the op-amp. This is highlighted in red. Recall that our op-amp takes the difference between the inputs and amplifies that difference at the output. Thus the op-amp performs both the difference operation between the input and feedback signals internally, and then amplifies the error (difference) signal to create the output.

To help analyze circuits of this kind, we will introduce two "golden rules" that could make our lives simpler.

18.4 Golden Rules

Much like we can use our KVL and KCL rules and Ohm’s law to solve circuits with sources and resistors, these op-amp golden rules help us solve circuits that include op-amps.
Recall the op-amp symbol and equivalent circuit, assuming $V_{SS} = -V_{DD}$. For simplicity, we will usually not draw the supply terminals in negative feedback circuits and assume that the output voltage is between whatever $V_{DD}$ and $V_{SS}$ we choose.

For an ideal op amp, i.e. an op-amp with internal gain approaching infinity, the "golden rules" are

- (1) **The current into both the input nodes is zero**, i.e. $I_+ = I_- = 0$. Consider the equivalent circuit for the op-amp above. Notice that there is no closed circuit connected to the positive or negative input terminal of the op-amp. Thus, no current can flow into the positive or negative input terminal. **This rule holds regardless of whether there is negative feedback or not.**

- (2) **When there is negative feedback, in steady state, the input and feedback voltages will be equal**, i.e. $V_+ = V_-$. When a circuit is first connected in negative feedback, there may be initially be some error signal. The feedback loop will try to correct this by changing the output voltage. This leads to a change in the feedback voltage, which leads to a change in the error signal, leading to another change in the output voltage, etc. Eventually, the error difference will go to zero, and the output will reach a steady state. **This rule only holds when there is negative feedback and holds only for large intrinsic op-amp gain, $A \to \infty$.** We will explore this in more depth in the next section. When we analyze circuits using this rule, we will assume that the op-amp gain is large and that the circuit has already reached steady state.

Now let’s use the golden rules to analyze the circuit we saw earlier:

First we apply KCL at the junction between $R_1$ and $R_2$ to get the following relationship

$$I_1 = I_2 + I_-.$$  

(2)
By the first golden rule, we know that $I_+ = I_- = 0$. Hence,

$$I_1 = I_2.$$  \hspace{1cm} (3)

Now let’s apply the second golden rule, $V_+ = V_-$. Using this, we have

$$V_{in} = V_{fb}.$$  \hspace{1cm} (4)

Now we can solve for $I_2$ using Ohm’s law, $V_{fb} = I_2 R_2$, hence

$$I_2 = \frac{V_{fb}}{R_2}.$$  \hspace{1cm} (5)

Using Ohm’s law on $R_1$, we also have $V_{out} - V_{fb} = I_1 R_1$. Hence,

$$I_1 = \frac{V_{out} - V_{fb}}{R_1}.$$  \hspace{1cm} (6)

However, we know that $I_1 = I_2$, which gives us the following relationship

$$I_1 = \frac{V_{out} - V_{fb}}{R_1} = \frac{V_{fb}}{R_2} = I_2,$$  \hspace{1cm} (7)

which is equivalent to

$$I_1 = \frac{V_{out} - V_{in}}{R_1} = \frac{V_{in}}{R_2} = I_2.$$  \hspace{1cm} (8)

Moving terms around, this gives us

$$V_{out} = V_{in} \left(1 + \frac{R_1}{R_2}\right).$$  \hspace{1cm} (9)

Notice that here the ratio $\frac{V_{out}}{V_{in}}$ only depends on the ratio $\frac{R_1}{R_2}$. This is a great property since it is rather difficult to produce resistors with a particular absolute resistance. Since $R_1$ and $R_2$ are always positive, the coefficient $1 + \frac{R_1}{R_2}$ is always positive. We call this the \textbf{non-inverting amplifier}.

\section*{18.5 Second golden rule revisited}

Recall that in the last section, we stated the second golden rule $V_+ = V_-$, i.e., the voltage potential at the positive input terminal (relative to ground) and the voltage at the negative input terminal (relative to the same ground) are the same when there is negative feedback. Now we would like to justify why this is the case. We return to the block diagram we drew earlier for a general negative feedback loop, but now focus on its application in op-amp circuits.

\begin{center}
\begin{tikzpicture}
    \node (input) [circle, draw, thick, fill=white] at (0,0) {$V_{in}$};
    \node (error) [circle, draw, thick, fill=white] at (1.5,0) {$V_{error}$};
    \node (amplifier) [circle, draw, thick, fill=white] at (3,0) {A};
    \node (output) [circle, draw, thick, fill=white] at (4.5,0) {$V_{out}$};
    \node (feedback) [circle, draw, thick, fill=white] at (3,-1.5) {f};

    \draw [->] (input) -- (error);
    \draw [->] (error) -- (amplifier);
    \draw [->] (amplifier) -- (output);
    \draw [->] (feedback) -- (output);
    \draw [->] (feedback) -- (amplifier);
    \draw [->] (input) -- (feedback);

\end{tikzpicture}
\end{center}
Let's build some intuition of what happens if something changes and how we may arrive back in steady state.

Suppose we are initially in steady state, and \( V_{in} \) suddenly increases. Since \( V_{error} = V_{in} - V_{fb} \), \( V_{error} \) will also increase. Then since \( A \) is a positive number, \( V_{out} \) also goes up, which causes \( V_{fb} \) to go up. This in turn causes the magnitude of \( V_{error} \) to go down, meaning that the system is going to stabilize itself. \(^1\)

Now what if we change the minus sign to a plus sign in the diagram, i.e., add the feedback signal instead of subtract it? This changes the system into a positive feedback system. With a similar logic, you could verify that if \( V_{in} \) goes up, \( V_{fb} \) goes up, but \( V_{error} \) goes up, which further causes \( V_{out} \) to go up. We see that it is not possible to stabilize the system.

Let's look at the negative feedback op-amp circuit we've seen earlier,

![Negative Feedback Op-Amp Circuit](image)

We know that when \( V_{in} \) increases, \( V_{out} \) also increases since \( V_{out} = A (V_{in} - V_{fb}) \). When \( V_{out} \) increases, \( V_{fb} = \frac{R_2}{R_1+R_2} V_{out} \) also increases, which then causes \( V_{error} \), and hence, \( V_{out} \) to go down. Relating this circuit back to the block diagram above, we can say that the feedback \( f = \frac{R_2}{R_1+R_2} \). Now let's derive why \( V_+ = V_- \) in this case. We know that in the above circuit, \( V_+ = V_{in} \) and \( V_- = V_{fb} \). Let's redraw the block diagram.

![Block Diagram](image)

Assume the system has reached steady state, so we can use the golden rules. We have

\[
V_{error} = V_+ - V_-
\]
\[
V_{out} = A V_{error} = A (V_+ - V_-)
\]
\[
V_- = f V_{out}
\]

\(^1\)This description implies some "delay" between each of the voltages changing. The impact of this delay, well as the values of \( A \) and \( f \), are explored more in EECS16B and EE128. Most feedback systems are designed with appropriate values of \( A \), \( f \), and delay so that the steady state occurs and we can use our golden rules.
Combining the last two equations, we have

\[ V_{out} = A (V_+ - fV_{out}) , \]  

which gives us

\[ V_{out} (1 + Af) = AV_+ . \]  

Finally, we have

\[ V_{out} = \frac{A}{1 + Af} V_+ . \]

Hence,

\[ V_- = fV_{out} = \frac{fA}{1 + Af} V_+ . \]

Now we know that the gain \( A \) is very large, hence \( fA \) is very large. Hence, the ratio

\[ \frac{fA}{1 + Af} \approx 1 . \]

Thus, when \( A \to \infty \), which is what we assume for an ideal op-amp, \( V_+ = V_- \).

### 18.6 Inverting Op-amp Amplifier

Let’s apply what we’ve learned so far about Golden rules and negative feedback to the following op-amp circuit:

Given an ideal op-amp (with power rails of sufficiently large magnitude), what is \( V_{out} \) if we input an arbitrary voltage of \( V_{in} \)?

The first golden rule says that \( I_- = I_+ = 0 \). Hence using KCL at the node labelled with voltage \( V_- \), we have

\[ I_{in} = I_- + I_f = 0 + I_f = I_f . \]  

\[ ^\text{2We will develop tools to determine if a circuit is in negative feedback in the next note. For now, let’s accept it as a given.} \]
We have

\[ I_{in} = I_f. \]  

(19)

Now, let’s apply the second golden rule, \( V_+ = V_- \). Since the positive input terminal is connected to ground, \( V_+ = 0 \). Hence, we have

\[ V_+ = V_- = 0. \]  

(20)

By Ohm’s law,

\[ I_{in} = \frac{V_{in} - V_-}{R_{in}} = \frac{V_{in}}{R_{in}} \]  

(21)

\[ I_f = \frac{V_- - V_{out}}{R_f} = -\frac{V_{out}}{R_f}. \]  

(22)

Since \( I_{in} = I_f \), we have

\[ \frac{V_{in}}{R_{in}} = -\frac{V_{out}}{R_f}. \]  

(23)

Moving terms around, we have

\[ V_{out} = -\frac{R_f}{R_{in}} V_{in}. \]  

(24)

Observe that the output voltage is a multiple of the input voltage with a scaling factor of \( -\frac{R_f}{R_{in}} \). Depending on chosen resistors, we can make this ratio anything between 0 and infinity. In addition, notice that \( V_{out} \) and \( V_{in} \) are of opposite signs. Since the output is amplified and inverted, this type of circuit is called an inverting amplifier.

### 18.7 Inverting Summing Amplifier

Now let’s take a look at a slightly more complicated op-amp circuit example with two voltage sources:
First, let’s apply the first golden rule, $I_- = I_+ = 0$. Applying KCL at the node labelled $V_-$, we have

$$I_1 + I_2 = I_- + I_3 = 0 + I_3.$$ (25)

Hence, we have

$$I_1 + I_2 = I_3.$$ (26)

Now by the second golden rule and the fact that the positive input terminal is connected to ground, we have

$$V_+ = V_- = 0.$$ (27)

Applying Ohm’s law at each of the three resistors, we have

$$I_1 = \frac{V_1 - V_-}{R_1} = \frac{V_1}{R_1}$$ (28)

$$I_2 = \frac{V_2 - V_-}{R_2} = \frac{V_2}{R_2}$$ (29)

$$I_3 = \frac{V_- - V_{out}}{R_3} = \frac{-V_{out}}{R_3}$$ (30)

Plugging in the above result to equation 26, we have

$$\frac{V_1}{R_1} + \frac{V_2}{R_2} = \frac{-V_{out}}{R_3}.$$ (32)

Multiplying both sides by $R_3$, we have

$$V_{out} = \frac{R_3}{R_1} V_1 - \frac{R_3}{R_2} V_2.$$ (33)

The output of this circuit $V_{out}$ is a sum of the two inputs $V_1$ and $V_2$, multiplied by the ratios $-\frac{R_3}{R_1}$ and $-\frac{R_3}{R_2}$. We call this circuit the **inverting summing amplifier**.

Note that we can also derive this final result from superposition. If we short $V_2$ and keep $V_1$ on, we get $V_{out} = \frac{R_3}{R_1} V_1$. Similarly, if we short $V_1$ and keep $V_2$ on, we get $V_{out} = \frac{-R_3}{R_2} V_2$. Adding these together, we get $V_{out} = \frac{-R_3}{R_1} V_1 - \frac{-R_3}{R_2} V_2$.

### 18.8 Practice Problems

These practice problems are also available in an interactive form on the course website.

1. True or False: We can use both Golden Rules to analyze any op-amp circuit.
2. True or False: An op-amp can operate without externally supplied power.
3. Find an expression for the output voltage $V_{\text{out}}$ in terms of the input voltage $V_{\text{in}}$ if the comparator is not railing (the output is between $V_{DD}$ and $V_{SS}$). Assume that the op-amp has a gain of $A$.

![Diagram of an op-amp circuit with resistors and ground symbols.]

4. An op-amp does not change the voltage of the circuit it is connected to because:

   (a) it has infinite gain.
   (b) it has infinite input resistance.
   (c) the Thévenin equivalent resistance at the output is 0.

5. If we switched the negative terminal and the positive terminal on an op-amp in negative feedback, would the gain across the amplifier be the same? For example, consider the op amp below.

![Diagram of an op-amp circuit with resistors and ground symbols.]

   (a) Yes, it would be the same.
   (b) No, it would be the inverse of the original gain.
   (c) No, because we can no longer assume that $V_+$ and $V_-$ are equal.

6. True or False: An ideal op-amp behaves as though it has infinite gain.