FIRST Name Freedum LAST Name Vektur SID (All Digits): 0123456789

- **(5 Points)** On *every* page, print legibly your name and ALL digits of your SID. For every page on which you do not write your name and SID, you forfeit a point, up to the maximum 5 points.

- **(10 Points) (Pledge of Academic Integrity)** Hand-copy, sign, and date the single-line text (which begins with *I have read, . . .*) of the Pledge of Academic Integrity on page 3 of this document. Your solutions will *not* be evaluated without this.

- **Urgent Contact with the Teaching Staff:** In case of an urgent matter, raise your hand if in-person, or send an email to `eecs16a@berkeley.edu` if online.

- **This document consists of pages numbered 1 through 14.** Verify that your copy of the exam is free of anomalies, and contains all of the specified number of pages. If you find a defect in your copy, contact the teaching staff immediately.

- This exam is designed to be completed within 70 minutes. However, you may use up to 80 minutes total—*in one sitting*—to tackle the exam.

  The exam starts at 8:10 pm California time. Your allotted window begins with respect to this start time. Students who have official accommodations of $1.5\times$ and $2\times$ time windows have 120 and 160 minutes, respectively.

- **This exam is closed book.** You may not use or access, or cause to be used or accessed, any reference in print or electronic form at any time during the exam, except one double-sided $8.5" \times 11"$ sheet of handwritten, original notes having no appendage.

  Collaboration is not permitted.

  Computing, communication, and other electronic devices (except dedicated timekeepers) must be turned off.

  Scratch paper will be provided to you; ask for more if you run out. You may not use your own scratch paper.

FIRST Name Freedum LAST Name Vektur SID (All Digits): 0123456789

- Please write neatly and legibly, because *if we can't read it, we can't evaluate it.*
- For each problem, limit your work to the space provided specifically for that problem. *No other work will be considered. For example, we will not evaluate scratch work. No exceptions.*
- Unless explicitly waived by the specific wording of a problem, you must explain your responses (and reasoning) succinctly, but clearly and convincingly.
- In some parts of a problem, we may ask you to establish a certain result—for example, "show this" or "prove that." Even if you're unable to establish the result that we ask of you, you may still take that result for granted—and use it in any subsequent part of the problem.
- If we ask you to provide a "reasonably simple expression" for something, by default we expect your expression to be in closed form—one *not* involving a sum $\sum$ or an integral $\int$—*unless* we explicitly tell you otherwise.
- Noncompliance with these or other instructions from the teaching staff—*including, for example, commencing work prematurely, or continuing it beyond the allocated time window*—is a serious violation of the Code of Student Conduct.

FIRST Name Freedum LAST Name Vektur SID (All Digits): 0123456789

Pledge of Academic Integrity

By my honor, I affirm that

(1) this document—which I have produced for the evaluation of my performance—reflects my original, bona fide work, and that I have neither provided to, nor received from, anyone excessive or unreasonable assistance that produces unfair advantage for me or for any of my peers;

(2) as a member of the UC Berkeley community, I have acted with honesty, integrity, respect for others, and professional responsibility—and in a manner consistent with the letter and intent of the campus Code of Student Conduct;

(3) I have not violated—nor aided or abetted anyone else to violate—the instructions for this exam given by the course staff, including, but not limited to, those on the cover page of this document; and

(4) More generally, I have not committed any act that violates—nor aided or abetted anyone else to violate—UC Berkeley, state, or Federal regulations, during this exam.

(10 Points) In the space below, hand-write the following sentence, verbatim. Then write your name in legible letters, sign, include your full SID, and date before submitting your work:

I have read, I understand, and I commit to adhere to the letter and spirit of the pledge above.

I have read, I understand, and I commit to adhere to the letter and spirit of the pledge above.

Full Name: Freedum Vektur Signature:

Date: Sep 17, 2024 Student ID: 0123456789

FIRST Name Freedum LAST Name Vektur SID (All Digits): 0123456789

Potentially Useful Facts That You May Use Without the Need to Prove Them:

- **Inner Product:** For every $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{C}^n$, we define $\langle \boldsymbol{x}, \boldsymbol{y} \rangle \triangleq \boldsymbol{x}^\mathsf{T} \boldsymbol{y}^* = \sum_{k=1}^{n} x_k \, y_k^*$.

  The complex conjugation may be omitted for vectors in $\mathbb{R}^n$—that is, for every $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$, we define $\langle \boldsymbol{x}, \boldsymbol{y} \rangle \triangleq \boldsymbol{x}^\mathsf{T} \boldsymbol{y} = \sum_{k=1}^{n} x_k \, y_k$.

- **Cauchy-Schwarz Inequality:** For all elements x and y in a vector space $\mathcal{V}$,

$$\left|\langle x, y \rangle\right| \leq ||x|| \, ||y||.$$

- **Triangle Inequality:** For all elements x and y in a vector space $\mathcal{V}$,

$$||x + y|| \leq ||x|| + ||y||.$$

- **Geometric Sum Formula** For all integers M and N, where $M \leq N$,

$$\sum_{\ell=M}^{N} \alpha^\ell = \begin{cases} \dfrac{\alpha^{N+1} - \alpha^M}{\alpha - 1} & \text{if } \alpha \neq 1 \\ N - M + 1 & \text{if } \alpha = 1. \end{cases}$$

- **Angle Between Vectors:** The angle θ between two *nonzero* elements x and y in a *real* vector space satisfies

$$\theta = \arccos \frac{\langle x, y \rangle}{||x|| \, ||y||}, \qquad \text{and} \qquad \cos\theta = \frac{\langle x, y \rangle}{||x|| \, ||y||}.$$

  Whether in a real or complex vector space, if $\langle x, y \rangle = 0$, we say x and y are orthogonal, and we denote this by $x \perp y$.

- **Sum of the first n positive integers:** $$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}.$$

- **Sum of the squares of the first n positive integers:** $$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.$$

- **Polynomials**
  - Any *nonzero* polynomial $p(t) = \sum_{k=0}^{n} a_k t^k$ in a real variable t, having real coefficients a_k, of degree $n \geq 0$, has exactly n roots, inclusive of multiplicity (i.e., root repetition)—real or complex.
  - Any polynomial $p(t) = \sum_{k=0}^{n} a_k t^k$ of degree $n \geq 0$ is infinitely differentiable—that is, it has derivatives of all orders.

FIRST Name Freedum LAST Name Vektur SID (All Digits): 0123456789

MT1.1 (35 Points) Consider a vector whose entries are the first n positive integers:

$$\boldsymbol{x} = \begin{bmatrix} 1 \\ 2 \\ \vdots \\ n \end{bmatrix}, \text{ where } n \in \{2, 3, 4, \ldots\}.$$

(a) (15 Points) Determine each of the following norms for $\boldsymbol{x}$. Each of your expressions must be in closed form—not left as a summation.

(i) (5 Points) The ℓ_∞-norm $||\boldsymbol{x}||_\infty$.

$$\|\underline{x}\|_\infty = \max(|x_1|, \ldots, |x_n|) = \max(1, \ldots, n) = n \Rightarrow \|\underline{x}\|_\infty = n$$

(ii) (5 Points) The ℓ_1-norm $||x||_1$.

$$\|\underline{x}\|_1 = |x_1| + \cdots + |x_n| = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

$$\|\underline{x}\|_1 = \frac{n(n+1)}{2}$$

(iii) (5 Points) The Euclidean (ℓ_2) norm $||\boldsymbol{x}||_2$.

$$\|\underline{x}\|_2 = \left(|x_1|^2 + \cdots + |x_n|^2\right)^{1/2} = \sqrt{1^2 + \cdots + n^2} \Rightarrow$$

$$\|\underline{x}\|_2 = \sqrt{\frac{n(n+1)(2n+1)}{6}}$$

FIRST Name Freedum LAST Name Vektur SID (All Digits): 0123456789

MT1.1 (Continued)

(b) (20 Points) Consider the vectors $\boldsymbol{y}$ and $\boldsymbol{z}$ defined as

$$\boldsymbol{y} = \begin{bmatrix}1\\2\\3\end{bmatrix} \qquad \text{and} \qquad \boldsymbol{z} = \begin{bmatrix}1\\2\\3\\4\end{bmatrix}.$$

$$\underline{y} = \begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix} \qquad \underline{z} = \begin{bmatrix}z_1\\z_2\\z_3\\z_4\end{bmatrix}$$

For each of these two vectors, construct a corresponding *nonzero* orthogonal vector—in particular, construct $\boldsymbol{y}_\perp$ and $\boldsymbol{z}_\perp$ such that

$$\langle \boldsymbol{y}, \boldsymbol{y}_\perp \rangle = 0, \qquad \text{and} \qquad \langle \boldsymbol{z}, \boldsymbol{z}_\perp \rangle = 0.$$

There is an uncountably-infinite set of answers for each. Construct only one for each, and explain (briefly) your thought process.

Method I:

$$\underline{y}_\perp = \begin{bmatrix}-1\\-1\\1\end{bmatrix}$$

We notice that $1+2=3$ (Sum of the first two components of $\underline{y}$ is equal to its third component). So, if we multiply each of the first two components by -1, and then add to the third component, we get zero.

$$\left\langle \underline{y}, \begin{bmatrix}-1\\-1\\1\end{bmatrix} \right\rangle = -y_1 - y_2 + y_3 = -1-2+3=0 \implies \underline{y}_\perp = \begin{bmatrix}-1\\-1\\1\end{bmatrix} \perp \underline{y}$$

Method II:

$$\underline{y}_\perp = \begin{bmatrix}-3\\0\\1\end{bmatrix} \text{ or } \begin{bmatrix}3\\0\\-1\end{bmatrix}$$

$$\langle \underline{y}, \underline{y}_\perp \rangle = \begin{bmatrix}1 & 2 & 3\end{bmatrix}\begin{bmatrix}3\\0\\-1\end{bmatrix} = 1\cdot 3 + \cancel{2\cdot 0}^{\,0} + 3(-1) = 3-3=0$$

Symmetrically reverse the entries of $\underline{y}$, flip sign on half of them. Since $\underline{y}$ has an odd number of components, make the middle entry of $\underline{y}_\perp$ (i.e., y_2) equal to zero.

Inspired by Method II, and noting that $\underline{z}$ has an even number of components, we reverse the entries, and flip sign on half of them.

That is:

$$\underline{z}_\perp = \begin{bmatrix} -z_4 \\ -z_3 \\ z_2 \\ z_1 \end{bmatrix} \Rightarrow \langle \underline{z}, \underline{z}_\perp \rangle = \begin{bmatrix} z_1 & z_2 & z_3 & z_4 \end{bmatrix} \begin{bmatrix} -z_4 \\ -z_3 \\ z_2 \\ z_1 \end{bmatrix}$$

$$= \cancel{-z_1 z_4} \cancel{-z_2 z_3} \cancel{+z_2 z_3} \cancel{+z_1 z_4} = 0$$

$$\Rightarrow \underline{z}_\perp \perp \underline{z}$$

FIRST Name Freedum LAST Name Vektur SID (All Digits): 0123456789

MT1.2 (60 Points) Throughout this problem—except for part numbering—i denotes $\sqrt{-1}$, and the Cartesian form of a complex number z is $z = a + ib$, where $a, b \in \mathbb{R}$.

(a) (40 Points) Express each of the following expressions in a Cartesian form.

(i) (10 Points) $\frac{1}{i} = \frac{1}{i} \times \frac{i}{i} = \frac{i}{-1} = -i$

(ii) (10 Points) $(1+i)^2 = (1+i)(1+i) = \cancel{1 \cdot 1}^{\,1} + \cancel{1 \cdot i}^{\,i} + \cancel{i \cdot 1}^{\,i} + \cancel{i \cdot i}^{\,-1} \Longrightarrow$

$(1+i)^2 = 2i$

(iii) (10 Points) i^4

$i^2 = -1$, $i^3 = i^2 i = -i$, $i^4 = -i(i) = 1$

In fact,

$i^{4k} = 1$, $i^{4k+1} = i$, $i^{4k+2} = -1$, and $i^{4k+3} = -i$

for all integers k.

FIRST Name Freedum LAST Name Vektur SID (All Digits): 0123456789

MT1.2 (Continued)

(iv) (10 Points) Show that $\sum_{k=0}^{3} i^k = 0$, and determine $\sum_{k=0}^{1000} i^k$ in Cartesian form.

$$\sum_{k=0}^{3} i^k = \frac{i^{3+1} - i^0}{i-1} = \frac{i^4 - 1}{i-1} = 0$$

$$\sum_{k=0}^{1000} i^k = \frac{i^{1001} - i^0}{i-1} = \frac{i-1}{i-1} = 1$$

FIRST Name Freedum LAST Name Vektur SID (All Digits): 0123456789

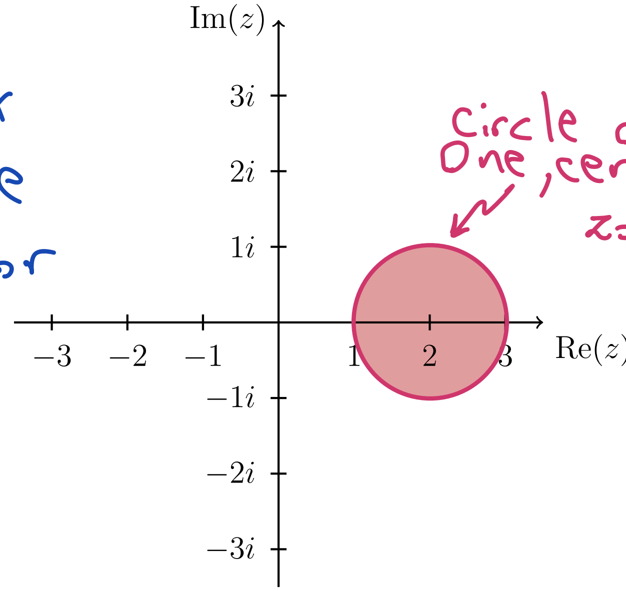
MT1.2 (Continued)

(b) (20 Points) For each of the following conditions, provide a well-labeled diagram of all points z in the complex plane that satisfy the stated condition.

Explain your work succinctly, and shade or otherwise highlight, the relevant region(s).

(i) (10 Points) $|2z - 4| \leq 2$. $\Longrightarrow |2(z-2)| \leq 2 \Rightarrow |z-2| \leq 1$

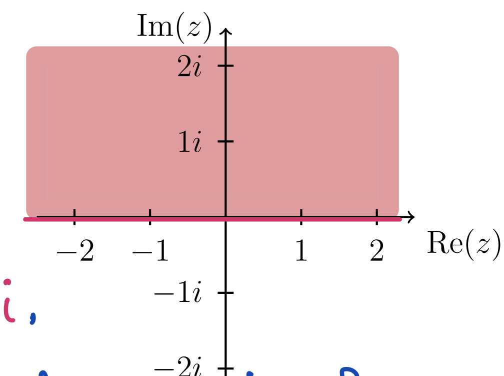
This is the set of points on the complex plane whose distance from 2 is less than, or equal to 1.



Circle of radius One, centered at $z=2$.

(ii) (10 Points) $|z - i| \leq |z + i|$.

This is the set of points on the complex plane that are no farther from i than they are from $-i$.



In other words, this is the set of points that are either

- closer to i than they are to $-i$ (i.e., are on the upper half plane); or
- equidistant to i and $-i$ (i.e., are on the Real axis).

FIRST Name Freedum LAST Name Vektur SID (All Digits): 0123456789

MT1.3 (35 Points) Consider the four vectors φ_k, where $k = 0, 1, 2, 3$, shown below:

$$\varphi_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \varphi_1 = \begin{bmatrix} 1 \\ i \\ -1 \\ -i \end{bmatrix}, \quad \varphi_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \quad \varphi_3 = \begin{bmatrix} 1 \\ -i \\ -1 \\ i \end{bmatrix}$$

(a) (5 Points) Show that φ_1 and φ_3 are orthogonal.

$$\langle \underline{\varphi}_1, \underline{\varphi}_3 \rangle = \underline{\varphi}_1^T \underline{\varphi}_3 = \begin{bmatrix} 1 & i & -1 & -i \end{bmatrix} \begin{bmatrix} 1 \\ i \\ -1 \\ -i \end{bmatrix} = 1^2 + \underbrace{\cancel{i^2}}_{-1} + \underbrace{\cancel{(-1)^2}}_{1} + \underbrace{\cancel{(-i)^2}}_{-1} = 0$$

$$\implies \langle \underline{\varphi}_1, \underline{\varphi}_3 \rangle = 0$$

(b) (10 Points) Determine $||\varphi_k||^2$ for all $k = 0, 1, 2, 3$.

Hint: You may determine the norm for one of the four vectors, and then provide a brief, convincing reason why the other vectors have the same norm.

We note that the magnitude-squared of every component of every vector in the set $\{\underline{\varphi}_0, \underline{\varphi}_1, \underline{\varphi}_2, \underline{\varphi}_3\}$ satisfies:

$$|1|^2 = |-1|^2 = |i|^2 = |-i|^2 = 1$$

So, $\|\underline{\varphi}_k\|^2 = 1 + 1 + 1 + 1 = 4$ for every $k \in \{0, 1, 2, 3\}$

$$\|\underline{\varphi}_k\|^2 = 1 \qquad \forall k \in \{0, 1, 2, 3\}$$

FIRST Name Freedum LAST Name Vektur SID (All Digits): 0123456789

MT1.3 (Continued)

(c) (20 Points) It turns out that all the four vectors φ_k, $k = 0, 1, 2, 3$ are mutually orthogonal—do *not* bother proving it here.

Assuming this mutual orthogonality, express the first canonical unit vector $\boldsymbol{e}_1$ in $\mathbb{R}^4$ as a linear combination of φ_k, $k = 0, 1, 2, 3$. That is, determine the coefficients α_k in

$$\boldsymbol{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \sum_{k=0}^{3} \alpha_k \varphi_k = \alpha_0 \varphi_0 + \alpha_1 \varphi_1 + \alpha_2 \varphi_2 + \alpha_3 \varphi_3.$$

$$\underline{e}_1 = \alpha_0 \underline{\varphi}_0 + \alpha_1 \underline{\varphi}_1 + \alpha_2 \underline{\varphi}_2 + \alpha_3 \underline{\varphi}_3$$

To determine α_ℓ, we take the inner product of each side with φ_ℓ, where $\ell = 0, 1, 2, 3$.

$$\langle \underline{e}_1, \underline{\varphi}_\ell \rangle = \left\langle \sum_{k=0}^{3} \alpha_k \underline{\varphi}_k, \underline{\varphi}_\ell \right\rangle = \sum_{k=0}^{3} \alpha_k \langle \underline{\varphi}_k, \underline{\varphi}_\ell \rangle$$

But $\underline{\varphi}_k \perp \underline{\varphi}_\ell$ for $k \neq \ell$, so all but one term on the right-hand side are eliminated (they're zero). We thus have

from the previous part

$$\langle \underline{e}_1, \underline{e}_\ell \rangle = \alpha_\ell \langle \underline{\varphi}_\ell, \underline{\varphi}_\ell \rangle = \alpha_\ell \|\underline{\varphi}_\ell\|^2 = 4\alpha_\ell \implies$$

$\alpha_\ell = \frac{1}{4} \langle \underline{e}_1, \underline{\varphi}_\ell \rangle$. But $\langle \underline{e}_1, \underline{\varphi}_\ell \rangle$ simply captures the first entry in $\underline{\varphi}_\ell$, and each first entry is 1. So, $\langle \underline{e}_1, \underline{\varphi}_\ell \rangle = 1$ for all $\ell \in \{0, 1, 2, 3\}$. $\implies \boxed{\alpha_\ell = \frac{1}{4} \; \forall \ell \in \{0, 1, 2, 3\}}$.

$$\alpha_\ell = \frac{1}{\|\varphi_\ell\|^2} \; \forall \ell \in \{0, 1, 2, 3\}$$

FIRST Name Freedum LAST Name Vektur SID (All Digits): 0123456789

MT1.4 (35 Points) Let $C^\infty(\mathbb{R})$ denote the vector space of functions of a continuous variable that are infinitely-differentiable over the entire real axis. A function belongs to this vector space if, and only if, it has derivatives of all orders—that is, all its derivatives are continuous. An element in this vector space is also called a C^∞ function (C^∞ is pronounced "C-infinity") or, easier yet, a *smooth* function.

Consider the finite set of smooth functions $\varphi_k : \mathbb{R} \to \mathbb{R}$, for $k = 0, 1, 2, \ldots, n$, where

$$\forall t \in \mathbb{R}, \qquad \varphi_0(t) = 1, \quad \varphi_1(t) = t, \quad \ldots \quad , \quad \varphi_k(t) = t^k, \quad \ldots \quad , \quad \varphi_n(t) = t^n$$

for some nonnegative integer n.

(a) (15 Points) Show that $\varphi_0, \ldots, \varphi_n$ are linearly independent.

Note: The zero element in $C^\infty(\mathbb{R})$ is the function that is zero for all $t \in \mathbb{R}$.

Construct the linear combination and set equal to zero for all t:

$$\alpha_0 \varphi_0(t) + \alpha_1 \varphi_1(t) + \cdots + \alpha_n \varphi_n(t) = \alpha_0 + \alpha_1 t + \cdots + \alpha_n t^n = 0 \quad \forall t \in \mathbb{R}$$

To establish linear independence, we must show that all the coefficients $\alpha_0, \ldots, \alpha_n$ must be zero (i.e., that $p(t) \triangleq \sum_{k=0}^{n} \alpha_k t^k$ is the zero polynomial.

Method I: We're told (in the formula sheet) that

– Any *nonzero* polynomial $p(t) = \sum_{k=0}^{n} a_k t^k$ in a real variable t, having real coefficients a_k, of degree $n \geq 0$, has exactly n roots, inclusive of multiplicity (i.e., root repetition)—real or complex.

This means that if a polynomial of degree $\leq n$ has more than n roots, it cannot be a nonzero polynomial (i.e., it must be the zero polynomial, which means $\alpha_0 = \alpha_1 = \cdots = \alpha_n = 0$).

Since $p(t) = \sum_{k=0}^{\infty} \alpha_k t^k = 0$ for all $t \in \mathbb{R}$, it has an uncountably infinite set of roots, so it must be the zero polynomial.

Method II:

We're told (in the formula sheet) that

– Any polynomial $p(t) = \sum_{k=0}^{n} a_k t^k$ of degree $n \geq 0$ is infinitely differentiable—that is, it has derivatives of all orders.

Looking at the polynomial equation

$$p(t) = \alpha_0 + \alpha_1 t + \dots + \alpha_n t^n = 0 \quad \forall t \in \mathbb{R}$$

Equality holds for all $t \in \mathbb{R}$, so it must hold for $t = 0$. Hence

$$p(0) = \alpha_0 = 0 \implies \alpha_0 = 0$$

Now differentiate $p(t)$ once:

$$\frac{dp(t)}{dt} = \alpha_1 + 2\alpha_2 t + \dots + n\alpha_n t^{n-1}$$

Since $p(t) = 0 \ \forall t \in \mathbb{R}$, then $\frac{dp(t)}{dt} = 0 \ \forall t \in \mathbb{R} \implies$

$$\left.\frac{dp(t)}{dt}\right|_{t=0} = 0 \implies \underline{\alpha_1 = 0}$$

Continuing in this manner, taking successively higher derivatives and setting the corresponding expression equal to zero, we can show that every coefficient $\alpha_k = 0$, $k \in \{0, 1, \dots, n\}$.

This implies that the $\varphi_k(t)$'s are linearly independent.

FIRST Name Freedum LAST Name Vektur SID (All Digits): 0123456789

MT1.4 (Continued)

(b) (20 Points) Let $\mathcal{P}_n$ denote a real-valued vector space of polynomials of degree less than, or equal to, n, where n is a nonnegative integer and $t \in \mathbb{R}$. You can think of $\mathcal{P}_n$ as the set constructed from all possible real linear combinations of $\varphi_k(t)$ for $k = 0, 1, \ldots, n$. A generic polynomial in $\mathcal{P}_n$ can be expressed as follows:

$$p(t) = \sum_{k=0}^{n} p_k\, \varphi_k(t) = \sum_{k=0}^{n} p_k\, t^k = \underbrace{\begin{bmatrix} p_0 & p_1 & \cdots & p_n \end{bmatrix}}_{\boldsymbol{p}^\top} \underbrace{\begin{bmatrix} 1 \\ t \\ \vdots \\ t^n \end{bmatrix}}_{\boldsymbol{f}(t)} = \boldsymbol{p}^\top \boldsymbol{f}(t),$$

where $\boldsymbol{f}(t) \in \mathbb{R}^{n+1}$ denotes the vector of monomials (you can think of $\boldsymbol{f}$ as a vector-valued function of the continuous variable t), $\boldsymbol{p} \in \mathbb{R}^{n+1}$ denotes the vector of the coefficients, and $^\top$ denotes transpose.

Define $\mathcal{V} \subseteq \mathcal{P}_n$ as the subset of all polynomials in $\mathcal{P}_n$ that have $t = 0$ <u>and</u> $t = 1$ as roots. That is,

$$\mathcal{V} = \left\{ v(t) = \sum_{k=0}^{n} v_k\, t^k \,\middle|\, v(0) = 0, v(1) = 0, v_k \in \mathbb{R}, k = 0, \ldots, n \right\}.$$

Show that $\mathcal{V}$ is a subspace.

Clearly, $\mathcal{V}$ is not empty. The zero polynomial is in $\mathcal{V}$.

So, we must establish only the closure axioms.

Let $v(t)$ and $w(t)$ denote polynomials in $\mathcal{V}$. Then

$v(0) = w(0) = v(1) = w(1) = 0$.

Let $y(t) = v(t) + w(t)$ $\forall t \in \mathbb{R}$. Then

$$\left.\begin{aligned} y(0) &= v(0) + w(0) = 0 + 0 = 0 \\ y(1) &= v(1) + w(1) = 0 + 0 = 0 \end{aligned}\right\} \Rightarrow y(t) \in \mathcal{V}$$

$\mathcal{V}$ is closed under addition

Now let $z(t) = \alpha v(t)$ for an arbitrary scalar $\alpha \in \mathbb{R}$.

$$\left.\begin{aligned} z(0) &= \alpha v(0) = 0 \\ z(1) &= \alpha v(1) = 0 \end{aligned}\right\} \Rightarrow z(t) \in \mathcal{V}$$

$\mathcal{V}$ is closed under scalar multiplication

FIRST Name Freedum LAST Name Vektur SID (All Digits): 0123456789

MT1.5 (20 Points) Let x and y denote two elements in a real vector space $\mathcal{V}$ defined over the scalar field $\mathbb{R}$. Assume that a norm $||\cdot||:\mathcal{V}\to\mathbb{R}$ has been defined on $\mathcal{V}$ (though not necessarily induced by an inner product).

Show that

$$-||x-y|| \le ||x|| - ||y|| \le ||x-y||.$$

Note: You may *not* assume that the vector space $\mathcal{V}$ is of a specific type (such as $\mathbb{R}^n$). You may *not* assume that the norm is of a specific type (such as the ℓ_2-norm). And you may *not* assume that an inner product has been defined on $\mathcal{V}$.

To receive full credit, you must resort only to the properties of a generic vector space and a valid, but otherwise unspecified, norm.

However, if you use a specific vector space (such as $\mathbb{R}^n$) in your reasoning; if you can't show the result without appeal to a specific norm; or if you assume a properly-defined inner product to argue your way to the result, you may receive a credit of at most 80% for this problem, provided your work contains no other shortcoming.

Hint: A carrier pigeon has told you that $x = (x-y)+y$—and, similarly, that $y = (y-x)+x$.

$x=(x-y)+y \implies \|x\| \le \|x-y\| + \|y\|$. Triangle Inequality

So,

$$\|x\| - \|y\| \le \|x-y\| \quad (*)$$

Similarly

$y=(y-x)+x \implies \|y\| \le \|y-x\| + \|x\|$. Triangle Inequality

So,

$-\|y-x\| \le \|x\| - \|y\|$.

But $\|y-x\| = \|x-y\|$. Scaling Property of Norms

So, $$-\|x-y\| \le \|x\| - \|y\| \quad (**)$$

Putting $(*)$ and $(**)$ together, we have the result

$-\|x-y\| \le \|x\| - \|y\| \le \|x-y\|$.

See further insight on the next page.

The result, which can be rewritten as follows,

$$\boxed{\big|\,\|x\| - \|y\|\,\big| \leq \|x-y\|}$$

is also known as the Reverse Triangle Inequality.

In terms of plane geometry, it simply states that

The length of any side of a triangle is greater than, or equal to, the difference between the lengths of the remaining two sides.