EECS 16A: Designing Information Devices and Systems I Midterm 2
Department of Electrical Engineering and Computer Sciences
UNIVERSITY OF CALIFORNIA, BERKELEY 29 October 2024

FIRST Name: QR LAST Name: U SID (All Digits): 1234567

- **(5 Points)** On *every* page, print legibly your name and ALL digits of your SID. For every page on which you do not write your name and SID, you forfeit a point, up to the maximum 5 points.
- **(10 Points) (Pledge of Academic Integrity)** Hand-copy, sign, and date the single-line text (which begins with *I have read, ...*) of the Pledge of Academic Integrity on page 3 of this document. Your solutions will *not* be evaluated without this.
- **Urgent Contact with the Teaching Staff:** In case of an urgent matter, raise your hand if in-person, or send an email to `eecs16a@berkeley.edu` if online.
- **This document consists of pages numbered 1 through 17.** Verify that your copy of the exam is free of anomalies, and contains all of the specified number of pages. If you find a defect in your copy, contact the teaching staff immediately.
- This exam is designed to be completed within 70 minutes. However, you may use up to 80 minutes total—*in one sitting*—to tackle the exam.

  The exam starts at 8:10 pm California time. Your allotted window begins with respect to this start time. Students who have official accommodations of $1.5\times$ and $2\times$ time windows have 120 and 160 minutes, respectively.
- **This exam is closed book.** You may not use or access, or cause to be used or accessed, any reference in print or electronic form at any time during the exam, except two double-sided $8.5"$ $\times$ $11"$ sheets of handwritten, original notes having no appendage.

  Collaboration is <u>not</u> permitted.

  Computing, communication, and other electronic devices (except dedicated timekeepers) must be turned off.

  Scratch paper will be provided to you; ask for more if you run out. You may not use your own scratch paper.
- Please write neatly and legibly, because *if we can't read it, we can't evaluate it.*
- For each problem, limit your work to the space provided specifically for that problem. *No other work will be considered. For example, we will not evaluate scratch work. No exceptions.*
- Unless explicitly waived by the specific wording of a problem, you must explain your responses (and reasoning) succinctly, but clearly and convincingly.
- In some parts of a problem, we may ask you to establish a certain result—for example, "show this" or "prove that." Even if you're unable to establish the result that we ask of you, you may still take that result for granted—and use it in any subsequent part of the problem.

- If we ask you to provide a "reasonably simple expression" for something, by default we expect your expression to be in closed form—one *not* involving a sum $\sum$ or an integral $\int$—*unless* we explicitly tell you otherwise.
- Noncompliance with these or other instructions from the teaching staff—*including, for example, commencing work prematurely, or continuing it beyond the allocated time window*—is a serious violation of the Code of Student Conduct.

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Pledge of Academic Integrity

By my honor, I affirm that

(1) this document—which I have produced for the evaluation of my performance—reflects my original, bona fide work, and that I have neither provided to, nor received from, anyone excessive or unreasonable assistance that produces unfair advantage for me or for any of my peers;

(2) as a member of the UC Berkeley community, I have acted with honesty, integrity, respect for others, and professional responsibility—and in a manner consistent with the letter and intent of the campus Code of Student Conduct;

(3) I have not violated—nor aided or abetted anyone else to violate—the instructions for this exam given by the course staff, including, but not limited to, those on the cover page of this document; and

(4) More generally, I have not committed any act that violates—nor aided or abetted anyone else to violate—UC Berkeley, state, or Federal regulations, during this exam.

(10 Points) In the space below, hand-write the following sentence, verbatim. Then write your name in legible letters, sign, include your full SID, and date before submitting your work:

I have read, I understand, and I commit to adhere to the letter and spirit of the pledge above.

I have read, I understand, and I commit to adhere to the letter and spirit of the pledge above.

Full Name: QR U Signature: QRU

Date: 29 OCT 2024 Student ID: 1234567

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Potentially Useful Facts That You May Use Without the Need to Prove Them:

- **Discrete Fourier Series (DTFS)**: Complex exponential Fourier series synthesis and analysis equations for a periodic discrete-time signal having period p:

$$x(n) = \sum_{k=\langle p \rangle} X_k\, e^{ik\omega_0 n} \qquad \longleftrightarrow \qquad X_k = \frac{1}{p} \sum_{n=\langle p \rangle} x(n)\, e^{-ik\omega_0 n}\,,$$

where $p = \dfrac{2\pi}{\omega_0}$ and $\langle p \rangle$ denotes a suitable discrete interval of length p (i.e., an interval containing p contiguous integers). For example, $\sum\limits_{k=\langle p \rangle}$ may denote $\sum\limits_{k=0}^{p-1}$ or $\sum\limits_{k=1}^{p}$.

- **Parseval's Identity for p-Periodic Discrete-Time Signals**: For a p-periodic discrete-time signal x that has DTFS coefficients X_k, the following identity holds:

$$\frac{1}{p} \sum_{n=\langle p \rangle} |x(n)|^2 = \sum_{k=\langle p \rangle} |X_k|^2.$$

This identity can also be written in vector form, using inner products. Let

$$\boldsymbol{x} = \begin{bmatrix} x(\ell) \\ \vdots \\ x(\ell + p - 1) \end{bmatrix} \qquad \text{and} \qquad \boldsymbol{X} = \begin{bmatrix} X_m \\ \vdots \\ X_{m+p-1} \end{bmatrix}$$

denote the vector of signal values (in one period) and the vector of DTFS coefficients, respectively. Then, Parseval's Identity can be written as follows:

$$\frac{1}{p} \langle \boldsymbol{x}, \boldsymbol{x} \rangle = \langle \boldsymbol{X}, \boldsymbol{X} \rangle.$$

- **Inverse of a 2×2 matrix**: The inverse of the matrix $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is given by $\mathbf{A}^{-1} = \dfrac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, provided that $ad - bc \neq 0$.

- **Some Trigonometric Values**:

$$\cos\left(\frac{\pi}{6}\right) = \sin\left(\frac{\pi}{3}\right) = \boxed{\sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2}} \qquad \sin\left(\frac{\pi}{6}\right) = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2} \qquad \cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2}.$$

$$\sin\left(\frac{\pi}{2}\right) = 1 \qquad \cos\left(\frac{\pi}{2}\right) = \cos\left(\frac{3\pi}{2}\right) = 0 \qquad \sin\left(\frac{3\pi}{2}\right) = -1.$$

- **Transposition and Inversion:** For any invertible matrix $\mathbf{A}$,

$$\left(\mathbf{A}^\mathsf{T}\right)^{-1} = \left(\mathbf{A}^{-1}\right)^\mathsf{T}.$$

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MT2.1 (40 Points) Rotation! Rotation! Rotation!

Consider the matrix

$$\mathbf{R}_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

(a) (7 Points) Determine a reasonably simple form for the matrix $\mathbf{R}_\theta^{-1}$.
You should be able to do this with little mathematical manipulation. But you must provide a succinct and clear explanation.

The matrix R_θ rotates any 2x2 vector $\underline{x}$ by angle θ to produce $\underline{y} = R_\theta \underline{x}$. The matrix R_θ^{-1} is the one that reverses the θ rotation—that is, the matrix that performs a $-\theta$ rotation:

$R_\theta^{-1} = R_{-\theta} = \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$. This can also be obtained by a blind application of the inverse formula for 2x2 matrices, but it misses the intuition behind the question.

(b) (8 Points) Determine the Gramian matrix $\mathbf{G}_{\mathbf{R}_\theta} = \mathbf{R}_\theta^\top \mathbf{R}_\theta$. What does your result say about the relationship between the columns of the matrix $\mathbf{R}_\theta$? What about the rows?

Note that $R_\theta^T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = R_\theta^{-1}$, so the Gramian matrix is $G_{R_\theta} = R_\theta^T R_\theta = R_\theta^{-1} R_\theta = I$.

The result shows that the columns of R_θ are orthonormal. To show this, let $R_\theta = [\underline{r}_1 \ \underline{r}_2]$, where $\underline{r}_1 = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$ and $\underline{r}_2 = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$.

$G_{R_\theta} = R_\theta^T R_\theta = \begin{bmatrix} \underline{r}_1^T \\ \underline{r}_2^T \end{bmatrix} [\underline{r}_1 \ \underline{r}_2] = \begin{bmatrix} \underline{r}_1^T \underline{r}_1 & \underline{r}_1^T \underline{r}_2 \\ \underline{r}_2^T \underline{r}_1 & \underline{r}_2^T \underline{r}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow$

$\underline{r}_1^T \underline{r}_1 = \underline{r}_2^T \underline{r}_2 = 1$ and $\underline{r}_1^T \underline{r}_2 = \underline{r}_2^T \underline{r}_1 = 0 \Rightarrow \{\underline{r}_1, \underline{r}_2\}$ orthonormal.

Using a similar argument on $R_\theta R_\theta^T = R_\theta R_\theta^{-1} = I$ we find that the rows, too, are orthonormal

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MT2.1 (Continued)

(c) (10 Points) Determine the matrix $\mathbf{R}_\theta^n$. Your expression for the entries of $\mathbf{R}_\theta^n$ must be in the simplest closed form possible.

Each time we apply R_θ to a vector, we rotate it by θ. So R_θ^n is simply a rotation by angle $n\theta$. Accordingly,

$$\boxed{R_\theta^n = R_{n\theta} = \begin{bmatrix} \cos(n\theta) & -\sin(n\theta) \\ \sin(n\theta) & \cos(n\theta) \end{bmatrix}}$$

(d) (15 Points) Consider the evolving vector

$$\boldsymbol{z}(n) = \left(\frac{1}{2}\right)^n \underbrace{\begin{bmatrix} \cos\left(\frac{\pi n}{2}\right) & -\sin\left(\frac{\pi n}{2}\right) \\ \sin\left(\frac{\pi n}{2}\right) & \cos\left(\frac{\pi n}{2}\right) \end{bmatrix}}_{R^n_{(\pi/2)}} \boldsymbol{z}(0) \qquad \text{for } n = 0, 1, 2, \ldots$$

$R_{(\pi/2)} = [$

Let $\boldsymbol{z}(0) = \begin{bmatrix} 16 \\ 0 \end{bmatrix}$.

Determine, and provide a well-labeled plot of, $\boldsymbol{z}(1)$, $\boldsymbol{z}(2)$, $\boldsymbol{z}(3)$, and $\boldsymbol{z}(4)$ on the two-dimensional plane. What is $\lim_{n\to\infty} \boldsymbol{z}(n)$?

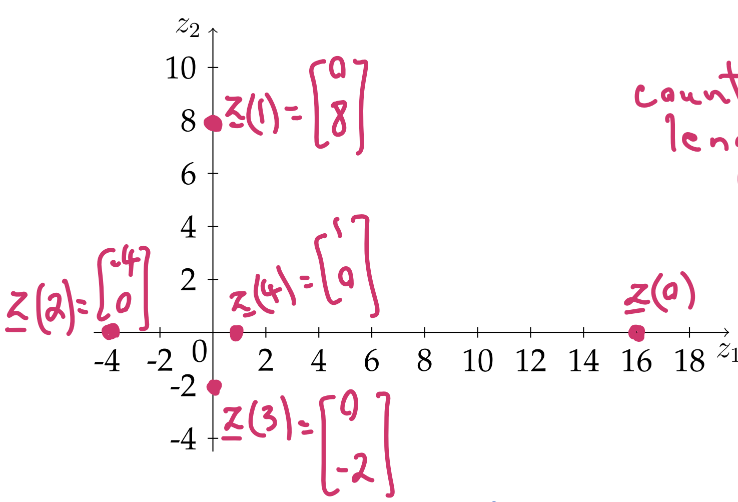
$\underline{z}(1) = \frac{1}{2} R_{(\pi/2)} \begin{bmatrix} 16 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \end{bmatrix}$

counterclockwise rotation by $\pi/2$, halved length $\rightarrow \begin{bmatrix} 0 \\ 8 \end{bmatrix}$

$\underline{z}(2) = \frac{1}{2} R_{(\pi/2)} \underline{z}(1) = \begin{bmatrix} -4 \\ 0 \end{bmatrix}$

$\underline{z}(3) = \frac{1}{2} R_{(\pi/2)} \underline{z}(2) = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$

$\underline{z}(4) = \frac{1}{2} R_{(\pi/2)} \underline{z}(3) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$



$\lim_{n\to\infty} \underline{z}(n) = \lim_{n\to\infty} \left(\frac{1}{2}\right)^n R^n_{(\pi/2)} \underline{z}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ b/c $\left(\frac{1}{2}\right)^n \rightarrow 0$ as $n \rightarrow \infty$.

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MT2.2 (40 Points) Triangular Matrices

This problem explores certain properties of triangular matrices. A square $n \times n$ matrix $\mathbf{L}$ is lower-triangular if each of its entries above the main diagonal is zero—that is, $\ell_{ij} = 0$ if $i - j < 0$, where $i, j \in \{1, 2, \ldots, n\}$. The following matrix $\mathbf{L}$ is lower-triangular:

$$\mathbf{L} = \begin{bmatrix} \ell_{11} & 0 & \cdots & 0 \\ \ell_{21} & \ell_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \ell_{n1} & \ell_{n2} & \cdots & \ell_{nn} \end{bmatrix}.$$

It can be proven that a lower-triangular matrix is invertible if, and only if, each of its diagonal entries ℓ_{ii} is nonzero, where $i \in \{1, 2, \ldots, n\}$.

You're not asked to prove this general result, but rather to use it in the parts below, as appropriate.

(a) (8 Points) A square $n \times n$ matrix $\mathbf{U}$ is upper-triangular if each of its entries below the main diagonal is zero—that is, a matrix whose entries satisfy $u_{ij} = 0$ if $i - j > 0$, where $i, j \in \{1, 2, \ldots, n\}$. The following matrix $\mathbf{U}$ is upper-triangular:

$$\mathbf{U} = \begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ 0 & u_{22} & \cdots & u_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & u_{nn} \end{bmatrix}.$$

Prove that $\mathbf{U}$ is invertible if, and only if, each of its diagonal entries u_{ii} is nonzero, where $i \in \{1, 2, \ldots, n\}$.

Note: To tackle this, use the result given in the problem stem—about lower-triangular matrices—plus any additional relevant property or properties of matrix inverses.

Any upper-triangular matrix U has a unique lower-triangular counterpart L, where the one-to-one correspondence is $U = L^T$.

So, $u_{ij} = \ell_{ji}$. Note that $u_{ii} = \ell_{ii}$ for all i, so U has all nonzero diagonal entries if, and only if, L has all nonzero diagonal entries.

L^{-1} exist $\iff \ell_{ii} \neq 0 \ \forall i$. Note that L^{-1} exists $\iff (L^{-1})^T$ exists.

But $(L^{-1})^T = (L^T)^{-1} = U^{-1}$.

So, U^{-1} exists $\iff (L^{-1})^T$ exists $\iff L^{-1}$ exists $\iff \ell_{ii} = u_{ii} \neq 0 \ \forall i$

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MT2.2 (Continued)

(b) (20 Points) Consider the lower-triangular 3×3 matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \\ 3 & 2 & 1 \end{bmatrix}.$$

A is lower-triangular, but $a_{22}=0$, so A is not invertible.

(i) (5 Points) Determine a basis for the nullspace $N(A)$.

We note that $\underline{a}_2 = 2\underline{a}_3 \Rightarrow 0\underline{a}_1 + \underline{a}_2 - 2\underline{a}_3 = \underline{0} \Rightarrow$ $\underline{z} = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$ spans $\mathcal{N}(A)$. It's clear that $\{\underline{a}_1, \underline{a}_3\}$ is linearly independent (since $\underline{a}_3 \neq \lambda \underline{a}_1$ for any scalar λ).

So, $\text{rank}(A) = 2 \Rightarrow$
$\dim \mathcal{N}(A) = 1$

$$\boxed{\mathcal{N}(A) = sp\left(\begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}\right)}$$

(ii) (5 Points) Determine a basis for the column space $C(A)$.

Since $\{\underline{a}_1, \underline{a}_3\}$ is linearly independent,

$$\boxed{C(A) = sp\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right)}$$

$\dim C(A) = \text{rank}\, A = 2$

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MT2.2 (b) (Continued)

(iii) (5 Points) Determine a basis for the left nullspace $N(A^T)$.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \\ 3 & 2 & 1 \end{bmatrix} = \begin{bmatrix} \underline{\alpha}_1^T \\ \underline{\alpha}_2^T \\ \underline{\alpha}_3^T \end{bmatrix} \Longrightarrow A^T = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \underline{\alpha}_1 & \underline{\alpha}_2 & \underline{\alpha}_3 \end{bmatrix}$$

$$\underline{\alpha}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \underline{\alpha}_2 = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} \quad \underline{\alpha}_3 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

$$\underline{\alpha}_2 = 2\underline{\alpha}_1 \Longrightarrow 2\underline{\alpha}_1 - \underline{\alpha}_2 + 0\underline{\alpha}_3 = \underline{0} \Longrightarrow A^T \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = \underline{0}$$

$$\boxed{\underline{\xi} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}}$$ left null vector

$\{\underline{\alpha}_1, \underline{\alpha}_3\}$ linearly independent (since one is not a scalar multiple of the other). So,

$$\boxed{\mathcal{N}(A^T) = \mathrm{sp}\left(\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}\right)}$$

(iv) (5 Points) Determine a basis for the row space $C(A^T)$.

Since $\{\underline{\alpha}_1, \underline{\alpha}_3\}$ is linearly independent, we have

$$\boxed{C(A^T) = \mathrm{sp}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}\right)}$$

Note that $\dim C(A^T) = \dim C(A) = 2 = \mathrm{rank}(A)$.

Also note that the null vector $\underline{z} = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} \perp \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \& \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$

So, $\mathcal{N}(A) \perp C(A^T)$ (nullspace $\perp$ row space).

Similarly, the left null vector $\underline{\xi} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \perp \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \& \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

So, $\mathcal{N}(A^T) \perp C(A)$, as expected.

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MT2.2 (Continued)

(c) (12 Points) Consider the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{bmatrix}.$$

Determine the matrix

$$\mathbf{B} = \begin{bmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{bmatrix}$$

such that $\mathbf{AB} = \mathbf{BA} = \mathbf{I}$, where $\mathbf{I}$ denotes the identity matrix.

You must evaluate each of the entries in the matrix $\mathbf{B}$.

$$AB = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{bmatrix}\begin{bmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{bmatrix} = \begin{bmatrix} a & 0 & 0 \\ 2a+3b & 3c & 0 \\ 4a+5b+6d & 5c+6e & 6f \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\boxed{a = 1}$ $\quad 3c = 1 \Rightarrow \boxed{c = 1/3}$ $\quad 6f = 1 \Rightarrow \boxed{f = 1/6}$

$\left.\begin{matrix} 2a+3b = 0 \\ a = 1 \end{matrix}\right\} \Rightarrow 2 + 3b = 0 \Rightarrow \boxed{b = -2/3}$

$\left.\begin{matrix} 5c+6e = 0 \\ c = 1/3 \end{matrix}\right\} \Rightarrow \frac{5}{3} + 6e = 0 \Rightarrow \boxed{e = -5/18}$

$\left.\begin{matrix} 4a+5b+6d = 0 \\ a = 1 \\ b = -2/3 \end{matrix}\right\} \Rightarrow 4 - \frac{10}{3} + 6d = 0 \Rightarrow \boxed{d = -1/9}$

$$\boxed{B = \begin{bmatrix} 1 & 0 & 0 \\ -2/3 & 1/3 & 0 \\ -1/9 & -5/18 & 1/6 \end{bmatrix}}$$

MT2.3 (35 Points)

Consider the 2×2 matrix

$$\mathbf{A} = \begin{bmatrix} 1 & -2 \\ 3 & -6 \end{bmatrix} = \begin{bmatrix} \underline{a}_1 & \underline{a}_2 \end{bmatrix} \qquad \underline{a}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \quad \underline{a}_2 = \begin{bmatrix} -2 \\ -6 \end{bmatrix}$$

(a) (5 Points) Explain why $\mathbf{A}$ is *not* invertible.

$\underline{a}_2 = -2\underline{a}_1 \implies$ A does not have linearly independent columns

Alternatively, we can say Row 2 is 3 x Row 1, so A does not have full row rank.

(b) (15 Points) We want to solve the equation $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$, where $\boldsymbol{b} = \begin{bmatrix} 1 \\ 12 \end{bmatrix}$.

Does a solution exist? If not, explain why. If yes, determine a solution.
Does the equation have a unique solution? If not, determine a second solution linearly independent of the first one.

$$C(A) = sp\left(\begin{bmatrix} 1 \\ 3 \end{bmatrix}\right) \qquad \underline{b} = \begin{bmatrix} 1 \\ 12 \end{bmatrix} \notin sp\left(\begin{bmatrix} 1 \\ 3 \end{bmatrix}\right)$$

Since $\underline{b}$ is <u>not</u> in the column space of A, the equation $A\underline{x} = \underline{b}$ has no solution.

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MT2.3 (Continued)

(c) (15 Points) We want to solve the equation $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$, where $\boldsymbol{b} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$.

Does a solution exist? If not, explain why. If yes, determine a solution.
Does the equation have a unique solution? If not, determine a second solution linearly independent of the first one.

$\underline{b} = 2\begin{bmatrix} 1 \\ 3 \end{bmatrix} = 2\underline{a}_1 \Rightarrow \underline{b} \in sp\left(\begin{bmatrix} 1 \\ 3 \end{bmatrix}\right) \Rightarrow$

The equation $A\underline{x} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$ has a solution— namely, $\boxed{\underline{x} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}}$: $\begin{bmatrix} 1 & -2 \\ 3 & -6 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \end{bmatrix} = \underline{b}$

However, the solution is not unique.

For example, we can add any nonzero scalar multiple of a null vector. Recall that

$\underline{a}_2 = -2\underline{a}_1 \Rightarrow 2\underline{a}_1 + \underline{a}_2 = \underline{0} \Rightarrow A\underbrace{\begin{bmatrix} 2 \\ 1 \end{bmatrix}}_{\underline{z}} = \underline{0}$

$\hat{\underline{x}} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} + \lambda\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is a solution for $\forall \lambda \in \mathbb{R}$.

For example, let $\lambda = 1 \Rightarrow \boxed{\hat{\underline{x}} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}}$ is a solution.

$A\hat{\underline{x}} = \begin{bmatrix} 1 & -2 \\ 3 & -6 \end{bmatrix}\begin{bmatrix} 4 \\ 1 \end{bmatrix} = 4\begin{bmatrix} 1 \\ 3 \end{bmatrix} + \begin{bmatrix} -2 \\ -6 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$ ✓

Alternatively, note that $\underline{b} = \begin{bmatrix} 2 \\ 6 \end{bmatrix} = -\begin{bmatrix} -2 \\ -6 \end{bmatrix} = -\underline{a}_2 \Rightarrow \boxed{\tilde{\underline{x}} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}}$ is a soln.

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MT2.4 (30 Points) QR Decomposition

Consider the 3×3 matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} \underline{a}_1 & \underline{a}_2 & \underline{a}_3 \end{bmatrix}$$

This problem explores the **QR** decomposition of **A**—that is, the factorization $\mathbf{A} = \mathbf{QR}$, where **Q** is an orthogonal matrix (such that $\mathbf{Q}^\top \mathbf{Q} = \mathbf{Q}\mathbf{Q}^\top = \mathbf{I}$), and **R** is upper-triangular—that is, each entry $r_{ij} = 0$ if $i - j > 0$, where $i, j \in \{1, 2, 3\}$.

(a) (18 Points) Show that the orthogonal matrix is $\mathbf{Q} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & -1 \\ 1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \end{bmatrix}$.

Hint: Apply the Gram-Schmidt Algorithm to the columns of **A** to determine the columns of $\boldsymbol{Q}$.

$$\underline{z}_1 = \underline{a}_1 \implies \underline{q}_1 = \frac{\underline{z}_1}{\|\underline{z}_1\|} = \frac{\underline{a}_1}{\|\underline{a}_1\|} \Rightarrow \boxed{\underline{q}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}$$

$$\boxed{\|\underline{a}_1\| = \sqrt{1^2 + 1^2 + 0^2} = \sqrt{2}}$$

$$\underline{z}_2 = \underline{a}_2 - \langle \underline{a}_2, \underline{q}_1 \rangle \underline{q}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \left(\frac{1}{\sqrt{2}}\right) \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Already of unit norm.

$$\boxed{\underline{q}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 0 \\ \sqrt{2} \end{bmatrix}}$$

$$\underline{z}_3 = \underline{a}_3 - \langle \underline{a}_3, \underline{q}_1 \rangle \underline{q}_1 - \langle \underline{a}_3, \underline{q}_2 \rangle \underline{q}_2$$

$$= \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \left(\frac{1}{\sqrt{2}}\right) \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1/2 \\ 0 \end{bmatrix} \implies \|\underline{z}_3\| = \sqrt{\frac{1}{4} + \frac{1}{4}} = \frac{1}{\sqrt{2}} \implies$$

$$\underline{q}_3 = \frac{1}{\|\underline{z}_3\|} \underline{z}_3 = \sqrt{2} \begin{bmatrix} -1/2 \\ 1/2 \\ 0 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \implies \boxed{\underline{q}_3 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}}$$

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MT2.4 (Continued)

(b) (12 Points) Determine the upper-triangular matrix $\mathbf{R}$.

Your answer must be in the simplest numerical closed form possible. Fractions—even involving irrational numbers—are fine, but decimals are not.

$$A = QR \implies Q^T A = \underbrace{\cancel{Q^T Q}}_{= I} R \implies R = Q^T A$$

$= I$ b/c Q is an orthogonal matrix.

$$R = Q^T A = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \\ -1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 2 & 2 & 1 \\ 0 & \sqrt{2} & \sqrt{2} \\ 0 & 0 & 1 \end{bmatrix}$$

$$\implies \boxed{R = \begin{bmatrix} \sqrt{2} & \sqrt{2} & 1/\sqrt{2} \\ 0 & 1 & 1 \\ 0 & 0 & 1/\sqrt{2} \end{bmatrix}}$$

Note that R is upper-triangular, as expected.

As a sanity check, compute QR:

$$QR = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 & -1 \\ 1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \end{bmatrix}\begin{bmatrix} \sqrt{2} & \sqrt{2} & 1/\sqrt{2} \\ 0 & 1 & 1 \\ 0 & 0 & 1/\sqrt{2} \end{bmatrix}$$

Absorb the $1/\sqrt{2}$ into R

$$= \begin{bmatrix} 1 & 0 & -1 \\ 1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1/2 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 0 & 1/2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

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MT2.5 (40 Points) DTFS

Consider a p-periodic discrete-time signal x—that is, $x : \mathbb{Z} \to \mathbb{R}$, where $x(n+p) = x(n)$ for all integers n and some positive integer p called a *period* of x. If p denotes the fundamental period, then $\omega_0 = 2\pi/p$ represents the fundamental frequency of the signal.

The Discrete-Time Fourier Series (DTFS) representation of the signal x is given by

$$x(n) = \sum_{k \in \langle p \rangle} X_k \, e^{ik\omega_0 n} \qquad \text{(Synthesis Equation)}$$

$$X_k = \frac{1}{p} \sum_{n \in \langle p \rangle} x(n) \, e^{-ik\omega_0 n} \qquad \text{(Analysis Equation)}$$

(a) (10 Points) Show that if a p-periodic real-valued discrete-time signal x is odd—$x(-n) = -x(n)$ for all integers n—then the DTFS coefficients

(i) (5 Points) X_k are odd in k—that is, $X_{-k} = -X_k$ for all $k \in \langle p \rangle$.
Hints: It's important that x is real-valued. Apply $-k$ to the Analysis Equation, and use a change of variables $m = -n$.

$$X_k = \frac{1}{p} \sum_{n \in \langle p \rangle} x(n) e^{-ik\omega_0 n} \Longrightarrow X_{-k} = \frac{1}{p} \sum_{n \in \langle p \rangle} x(n) e^{ik\omega_0 n} \quad \text{let } m=-n \Longrightarrow$$

$$X_{-k} = \frac{1}{p} \sum_{m \in \langle p \rangle} x(-m) e^{-ik\omega_0 m} \quad x(-m) = -x(m) \Longrightarrow X_{-k} = \frac{-1}{p} \sum_{m \in \langle p \rangle} x(m) e^{-ik\omega_0 m} = -X_{-k}$$

$$\Longrightarrow \boxed{X_{-k} = -X_k}$$

(ii) (5 Points) X_k are *purely imaginary*—of the form $X_k = ib$ for some real value b.
Hint: A complex number z is purely imaginary if, and only if, $z^* = -z$. To see this, let $z = a + ib$. Then $z^* = a - ib$, and $-z = -a - ib$. Clearly, $z^* = -z$ if, and only if, $a = -a = 0$, which means $z = ib$—purely imaginary.

$$X_k^* = \frac{1}{p} \sum_{n \in \langle p \rangle} x^*(n) e^{ik\omega_0 n} \quad x(n) \in \mathbb{R}, \forall n \Longrightarrow X_k^* = \frac{1}{p} \sum_{n \in \langle p \rangle} x(n) e^{ik\omega_0 n} \quad \text{let } m=-n \Longrightarrow$$

$$X_k^* = \frac{1}{p} \sum_{m \in \langle p \rangle} x(-m) e^{-ik\omega_0 m} \quad x(-m) = -x(m) \Longrightarrow X_k^* = \frac{-1}{p} \sum_{m \in \langle p \rangle} x(m) e^{-ik\omega_0 m} = -X_k \Longrightarrow$$

$$\boxed{X_k^* = -X_k} \Longrightarrow \boxed{X_k \text{ is purely imaginary}}$$

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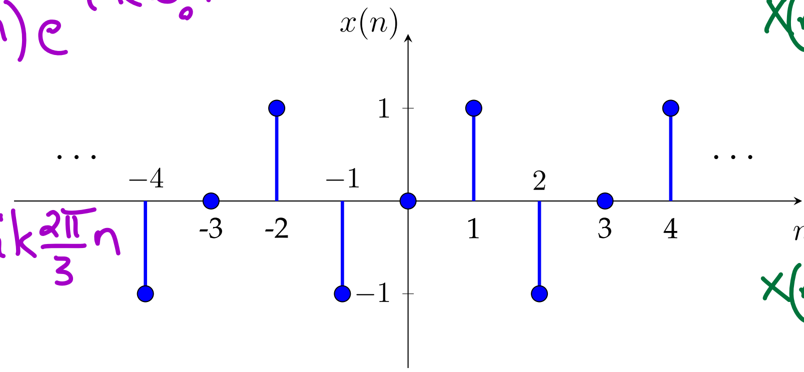
MT2.5 (Continued)

(b) (30 Points) Consider the discrete-time periodic signal shown below.

$$X_k = \frac{1}{p}\sum_{n\in\langle p\rangle} x(n)e^{-ik\omega_0 n} \qquad x(n) = \sum_{k\in\langle p\rangle} X_k e^{ik\omega_0 n}$$



$$X_k = \frac{1}{3}\sum_{n=-1}^{1} x(n)e^{-ik\frac{2\pi}{3}n} \qquad x(n) = \sum_{k=-1}^{1} X_k e^{ik\frac{2\pi}{3}n}$$

The signal has fundamental period $p = 3$. $\Longrightarrow \omega_0 = \frac{2\pi}{p} = \frac{2\pi}{3}$

(i) (15 Points) Determine all the DTFS coefficients X_k for the signal x.

$$X_0 = \frac{1}{p}\sum_{n\in\langle p\rangle} x(n)$$ average value of the signal over one period

$$X_0 = \frac{1}{3}\sum_{n=-1}^{1} x(n) = \frac{x(-1)+x(0)+x(1)}{3} = \frac{-1+0+1}{3} = 0 \Longrightarrow \boxed{X_0 = 0}$$

$$X_1 = \frac{1}{3}\sum_{n=-1}^{1} x(n)e^{-i\frac{2\pi}{3}n} = \frac{1}{3}\left[\underbrace{x(-1)}_{-1}e^{i\frac{2\pi}{3}} + \underbrace{x(0)}_{0} + \underbrace{x(1)}_{1}e^{-i\frac{2\pi}{3}}\right]$$

$$X_1 = \frac{-2i}{3}\,\frac{e^{i\frac{2\pi}{3}} - e^{-i\frac{2\pi}{3}}}{2i} = \frac{-2i}{3}\sin\left(\frac{2\pi}{3}\right) = \frac{-\cancel{2}i}{3}\,\frac{\sqrt{3}}{\cancel{2}} = \frac{-i}{\sqrt{3}}$$

$$X_{-1} = -X_1 = \frac{i}{\sqrt{3}}.$$ So

$$\boxed{X_{-1} = \frac{i}{\sqrt{3}},\ X_0 = 0,\ X_1 = \frac{-i}{\sqrt{3}}}$$

MT2.5 (b) (Continued)

(ii) (8 Points) Show that the signal x can be described by $x(n) = \alpha \sin(\beta n)$ for all integers n, and for some parameters α and β.

Determine α and β numerically. Your answers must be in the simplest form possible, but not expressed as decimals.

$$x(n) = \sum_{k\in\langle p\rangle} X_k e^{ik\omega_0 n} = \sum_{k=-1}^{1} X_k e^{ik\frac{2\pi}{3}n}$$

$$x(n) = X_{-1} e^{-i\frac{2\pi}{3}n} + \cancelto{0}{X_0} + X_1 e^{i\frac{2\pi}{3}n} = \frac{i}{\sqrt{3}} e^{-i\frac{2\pi}{3}n} - \frac{i}{\sqrt{3}} e^{i\frac{2\pi}{3}n}$$

$$= \frac{-i(2i)}{\sqrt{3}}\left(\frac{e^{i\frac{2\pi}{3}n} - e^{-i\frac{2\pi}{3}n}}{2i}\right) = \underbrace{\frac{2}{\sqrt{3}}}_{\alpha}\sin\left(\underbrace{\frac{2\pi}{3}}_{\beta} n\right)$$

$$\alpha = \frac{2}{\sqrt{3}} \qquad \beta = \frac{2\pi}{3}$$

(iii) (7 Points) Evaluate $\langle \mathbf{X}, \mathbf{X}\rangle$ where $\boldsymbol{X}$ denotes the vector of all the DTFS coefficients.

$$\langle \underline{X},\underline{X}\rangle = \sum_{k\in\langle p\rangle} |X_k|^2 = \frac{1}{p}\langle \underline{x},\underline{x}\rangle = \frac{1}{p} \sum_{n\in\langle p\rangle}|x(n)|^2$$

By Parseval's

$$\langle \underline{X},\underline{X}\rangle = \frac{1}{3}\sum_{n=-1}^{1}|x(n)|^2 = \frac{\cancelto{1}{|x(-1)|^2} + \cancelto{0}{|x(0)|^2} + \cancelto{1}{|x(1)|^2}}{3} = \frac{2}{3}$$

$$\boxed{\langle \underline{X},\underline{X}\rangle = \frac{2}{3}}$$

This provides an opportunity to run a sanity check on our X_{-1}, X_0, X_1 in the previous part: $\langle \underline{X},\underline{X}\rangle = \cancelto{1/3}{|X_{-1}|^2} + \cancelto{0}{|X_0|^2} + \cancelto{1/3}{|X_1|^2} = \frac{2}{3}$