Pledge of Academic Integrity

By my honor, I affirm that

(1) this document—which I have produced for the evaluation of my performance—reflects my original, bona fide work, and that I have neither provided to, nor received from, anyone excessive or unreasonable assistance that produces unfair advantage for me or for any of my peers;

(2) as a member of the UC Berkeley community, I have acted with honesty, integrity, respect for others, and professional responsibility—and in a manner consistent with the letter and intent of the campus Code of Student Conduct;

(3) I have not violated—nor aided or abetted anyone else to violate—the instructions for this exam given by the course staff, including, but not limited to, those on the cover page of this document; and

(4) More generally, I have not committed any act that violates—nor aided or abetted anyone else to violate—UC Berkeley, state, or Federal regulations, during this exam.

(10 Points) In the space below, hand-write the following sentence, verbatim. Then write your name in legible letters, sign, include your full SID, and date before submitting your work:

I have read, I understand, and I commit to adhere to the letter and spirit of the pledge above.

____

____

Full Name: ____________________ Signature: ____________________

Date: ____________________ Student ID: ____________________

MT3.1 (30 Points) Hide and Seek

Members of 16A course staff are playing hide and seek in Cory Hall! In an effort to win, they try various different strategies to mask their coordinates.

(a) (15 points) Melissa tries encrypting her coordinates with the following setup, where her real coordinates are $\mathbf{x} \in \mathbb{R}^2$ and her encrypted coordinates are $\mathbf{z} \in \mathbb{R}^2$ by a change of basis. In other words, we can represent the relation between the $\mathbf{x}$ and $\mathbf{z}$ as:

$$\mathbf{x} = \mathbf{C}\mathbf{z}$$

Melissa first starts at the origin in the standard basis, with $\mathbf{x}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. She first moves one unit north in the standard basis to $\mathbf{x}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and her new encrypted position is $\mathbf{z}_1 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$. Starting from this new position, she then moves east one unit in the standard basis to $\mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and her final encrypted position is $\mathbf{z}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

What is the change of basis matrix $\mathbf{C}$ mapping her coordinates from the encrypted basis to the standard basis?

Solution: Let's set up a system of equations based on what is given in the problem! We have that:

$$\mathbf{C}\begin{bmatrix}-1\\0\end{bmatrix} = \begin{bmatrix}0\\1\end{bmatrix}$$

$$\mathbf{C}\begin{bmatrix}-1\\1\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix}$$

Letting $\mathbf{C} = \begin{bmatrix}a & b\\c & d\end{bmatrix}$, we get the system of equations $-a = 0$, $-c = 1$, $-a + b = 1$, $-c + d = 1$. Therefore we have that

$$\mathbf{C} = \begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix}$$

(b) (15 points) Melissa decides to switch her strategy. She encrypts her coordinates with a new unknown change of basis matrix $\mathbf{T}$, but shares the eigenvectors for the matrix:

$$\mathbf{v}_1 = \begin{bmatrix}-1\\1\end{bmatrix},\ \mathbf{v}_2 = \begin{bmatrix}1\\1\end{bmatrix}$$

Athul, a seeker, figures out the eigenvalues $\lambda_1 = -1$ for $\mathbf{v}_1$ and $\lambda_2 = 5$ for $\mathbf{v}_2$. Given Melissa's encrypted coordinates are $\mathbf{z}_c = \begin{bmatrix}0\\4\end{bmatrix}$, find her real position.

Hint: Write $\mathbf{z}_c$ as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$. What happens when you apply $\mathbf{T}$ to this linear combination?

Solution: Melissa's real position is computed with $\mathbf{T}\mathbf{z}_c$, where $\mathbf{T}$ is a matrix mapping from the encrypted to the standard basis. Luckily, we don't need to matrix multiply! Representing $\mathbf{z}_c$ as a linear combination of the given eigenvectors, we can rewrite $\mathbf{T}\mathbf{z}_c$ as $\lambda_1\alpha_1\mathbf{v}_1 + \lambda_2\alpha_2\mathbf{v}_2$. Plugging in our values, we solve $-1 \cdot 2 \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} + 5 \cdot 2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 12 \\ 8 \end{bmatrix}$

MT3.2 (45 Points) Ridge Regression

(a) (15 points) Consider the matrix vector equation $\mathbf{A}\mathbf{x} = \mathbf{b}$ where

$$\mathbf{A} = \begin{bmatrix} 2 & 2 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Solve for $\hat{\mathbf{x}}$ using least squares.

Solution: Applying the least squares equation, we have:

$$\begin{aligned}
(\mathbf{A}^\top \mathbf{A})^{-1}\mathbf{A}^\top \mathbf{b} &= (\begin{bmatrix} 2 & 2 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}^\top \begin{bmatrix} 2 & 2 \\ -1 & 2 \\ 1 & 1 \end{bmatrix})^{-1} \begin{bmatrix} 2 & 2 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}^\top \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \\
&= (\begin{bmatrix} 6 & 3 \\ 3 & 9 \end{bmatrix})^{-1} \begin{bmatrix} 3 \\ 9 \end{bmatrix} \\
&= \frac{1}{45} \begin{bmatrix} 9 & -3 \\ -3 & 6 \end{bmatrix} \begin{bmatrix} 3 \\ 9 \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ 1 \end{bmatrix}
\end{aligned}$$

(b) (10 points) Now consider a different $\mathbf{A}$ matrix

$$\mathbf{A} = \begin{bmatrix} 2 & -4 \\ 1 & -2 \\ -3 & 6 \end{bmatrix}$$

In two sentences or less, explain why the traditional least squares formula won't work for this matrix.

Solution: By inspection, the columns of $\mathbf{A}$ are linearly dependent, so $\mathbf{A}^\top\mathbf{A}$ is not invertible.

It turns out that we can still run least squares with this matrix through a variant called **ridge regression**. Traditional least squares solves the following optimization problem

$$\min_{\mathbf{x}} ||\mathbf{A}\mathbf{x} - \mathbf{b}||^2$$

Ridge regression adds what's called an *L2 regularization* term to this objective function

$$\min_{\mathbf{x}} ||\mathbf{A}\mathbf{x} - \mathbf{b}||^2 + ||\eta\mathbf{x}||^2, \eta > 0 \in \mathbb{R}$$

The resulting ridge regression solution for $\hat{\mathbf{x}}$ is

$$\hat{\mathbf{x}} = (\mathbf{A}^\top\mathbf{A} + \eta\mathbf{I})^{-1}\mathbf{A}^\top\mathbf{b}$$

In the next couple of parts we will motivate one of the benefits of L2 regularization.

(c) (10 points) Assume the eigenvalues of $\mathbf{A}^\top\mathbf{A}$ are $\lambda_1 = 5$ and $\lambda_2 = 0$. Show that the eigenvalues of $\mathbf{A}^\top\mathbf{A} + \eta\mathbf{I}$ are $5 + \eta$ and η. *HINT: $\mathbf{A}^\top\mathbf{A} + \eta\mathbf{I}$ and $\mathbf{A}^\top\mathbf{A}$ have the same eigenvectors.*

Solution:

Solution 1:

Note that $\mathbf{I} = \mathbf{VIV}$.

Rewriting the matrix expression with the diagonalization of $\mathbf{A}^\top \mathbf{A}$, we get:

$$\begin{aligned}
\mathbf{A}^\top \mathbf{A} + \eta \mathbf{I} &= \mathbf{V}\mathbf{\Lambda}\mathbf{V}^\top + \eta \mathbf{I} \\
&= \mathbf{V}\mathbf{\Lambda}\mathbf{V}^\top + \eta \mathbf{V}\mathbf{V}^\top \\
&= \mathbf{V}(\mathbf{\Lambda}\mathbf{V}^\top + \eta \mathbf{V}^\top) \\
&= \mathbf{V}(\mathbf{\Lambda} + \eta \mathbf{I})\mathbf{V}^\top \\
&= \mathbf{V}\left(\begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} \eta & 0 \\ 0 & \eta \end{bmatrix}\right)\mathbf{V}^\top \\
&= \mathbf{V}\begin{bmatrix} 5+\eta & 0 \\ 0 & 0+\eta \end{bmatrix}\mathbf{V}^\top
\end{aligned}$$

Therefore, $\lambda_1 = 5 + \eta$ and $\lambda_2 = \eta$

Solution 2: Suppose that $\mathbf{v}_5$ is the eigenvector corresponding to $\lambda = 5$. Then we have that:

$$\begin{aligned}
(\mathbf{A}^\top \mathbf{A} + \eta \mathbf{I})\mathbf{v}_5 &= \mathbf{A}^\top \mathbf{A}\mathbf{v}_5 + \eta \mathbf{v}_5 \\
&= 5\mathbf{v}_5 + \eta \mathbf{v}_5 \\
&= (5+\eta)\mathbf{v}_5
\end{aligned}$$

By the definition of eigenvalues, we have that $5 + \eta$ is an eigenvalue of $\mathbf{A}^\top \mathbf{A} + \eta \mathbf{I}$. Similarly, applying this argument to $\lambda = 0$ gives us $\eta + 0 = \eta$ is also an eigenvalue of $\mathbf{A}^\top \mathbf{A} + \eta \mathbf{I}$.

(d) (10 points) Suppose that in the general case, we have that the eigenvalues of $\mathbf{A}^\top \mathbf{A}$ are $\lambda_1, \lambda_2 \ldots \lambda_n$. Recall that η is strictly greater than 0. Argue that all the eigenvalues of $\mathbf{A}^\top \mathbf{A} + \eta \mathbf{I}$ must be strictly greater than 0. You may reference the results of the last part even if you did not do the problem. What does this say about the invertibility of $\mathbf{A}^\top \mathbf{A} + \eta \mathbf{I}$?

Solution: By the spectral theorem, we know that that eigenvalues of $\mathbf{A}^\top \mathbf{A}$ are always real. Furthermore, the matrix $\mathbf{A}^\top \mathbf{A}$ is a positive semidefinite matrix which means that its eigenvalues are always nonnegative i.e. $\lambda_i \geq 0$.

This next derivation is not required to receive full credit but is included for completeness. Let $\mathbf{v}$ be an eigevector of $\mathbf{A}^\top\mathbf{A}$ with eigenvalue λ_i, then

$$\begin{aligned}
\mathbf{v}^\top\mathbf{A}^\top\mathbf{A}\mathbf{v} &= (\mathbf{A}\mathbf{v})^\top\mathbf{A}\mathbf{v} \\
&= \|\mathbf{A}\mathbf{v}\|^2 \geq 0 \\
\mathbf{v}^\top\mathbf{A}^\top\mathbf{A}\mathbf{v} &= \lambda_i\mathbf{v}^\top\mathbf{v} \\
&= \lambda_i\|\mathbf{v}\|^2 \\
&\implies \lambda_i\|\mathbf{v}\|^2 \geq 0
\end{aligned}$$

Since $\mathbf{v} \neq \mathbf{0}$, we have that $\lambda_i \geq 0$.

Also, as given in the objective function for ridge regression, $\eta > 0$. As seen in the previous part, the eigenvalues of $\mathbf{A}^\top\mathbf{A} + \eta\mathbf{I}$ are given by $\eta + \lambda_i$, meaning the eigenvalues of $\mathbf{A}^\top\mathbf{A} + \eta\mathbf{I}$ are strictly greater than zero.

Since the eigenvalues of $\mathbf{A}^\top\mathbf{A} + \eta\mathbf{I}$ are nonzero, its nullspace is trivial. Therefore, its inverse exists. This also means that the ridge regression solution always exists no matter the rank of $\mathbf{A}$!

MT3.3 (45 Points) Robot Grasping with PCA

As a robotics automation engineer, we would like to develop a robotic system that can pick up ellipse-shaped objects from a table, using only an overhead camera image as input.

In order to estimate the pose of our elliptical objects, we will characterize it by their center point (x, y), while their orientation is described by a unit vector pointing in the direction of their major axis (the longer axis).

To derive these positions, we will sample points in the image that appear to lie on the object. By doing so uniformly across the image, we assume that they are roughly representative of the object's actual position. An example plot is as follows:

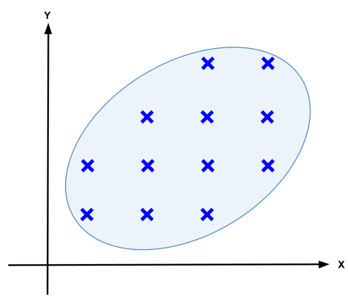

Figure 1: Example of Object Sampling

For one particular sampled object, assume that we store the (x, y) positions of our sampled points as **row data** in the following matrix:

$$\mathbf{A} = \begin{bmatrix} 3 & 2 \\ 1 & 1 \\ 2 & 2 \\ 3 & 3 \\ 1 & 2 \end{bmatrix}$$

(a) (5 points) Given the data matrix $\mathbf{A}$, estimate the position $(\bar{x}, \bar{y})$ of the center of the ellipse.

Solution:
Given only the data in the matrix, the best estimate for the ellipse's center is simply the average of the samples. This can be computed by finding the vector averages of the columns of $\mathbf{A}$, or simply the average x and y coordinates of our points. Mathematically, we can see that

$$\bar{x} = \frac{3+1+2+3+1}{5} = \frac{10}{5} = 2$$

$$\bar{y} = \frac{2+1+2+3+2}{5} = \frac{10}{5} = 2$$

Therefore, the estimated center of our ellipse is $(2, 2)$.

(b) (15 points) Ignoring the previous subpart, imagine that the data has now been centered about the origin, and can now be characterized by the following data matrix $\mathbf{D}$.

$$\mathbf{D} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ -2 & 2 \\ 2 & -2 \\ -1 & -1 \end{bmatrix}$$

Given this matrix, determine the best estimate for the unit vector $\hat{\mathbf{m}}$ that characterizes the direction of the ellipse's major axis. This axis should correspond to the **direction of maximum variance** in our sampled points.

Solution:

Since our data is stored in rows, the major axis vector should be our principal right singular vector, or the first row of our $\mathbf{V}^\top$ matrix in the SVD of $\mathbf{D}$.

Rather than necessarily computing the complete SVD, we can instead simply compute $\mathbf{D}^\top\mathbf{D}$ and find the eigenvectors to construct $\mathbf{V}^\top$. Making sure that $\mathbf{V}^\top$ is orthonormal, the correct orientation vector $\hat{\mathbf{m}}$ will simply be the first row.

Mathematically, we find that

$$\mathbf{D}^\top\mathbf{D} = \begin{bmatrix} 1 & 0 & -2 & 2 & -1 \\ 1 & 0 & 2 & -2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ -2 & 2 \\ 2 & -2 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 10 & -6 \\ -6 & 10 \end{bmatrix}$$

By inspection, we can see that:

- $\begin{bmatrix} 10 & -6 \\ -6 & 10 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 16 \\ -16 \end{bmatrix} = 16 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$
- $\begin{bmatrix} 10 & -6 \\ -6 & 10 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

so the eigenvectors of $\mathbf{D}^\top\mathbf{D}$ are $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ with eigenvalues 16 and 4 (this could also be discovered via the characteristic polynomial).

The vector with the largest eigenvalue (and thus largest singular value) is $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Normalizing tells us that

$$\hat{\mathbf{m}} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$$

Now that we have successfully estimated the pose of our target object, we can actually pick it up using a gripper as shown below!

(c) (5 points) Ignoring the previous subpart, assume we found that $\hat{\mathbf{m}} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$ is the direction of maximum variance. In order to maximize our chances of successfully grasping the object, however, we want to grasp along the **shortest** axis of our ellipse estimation. Provide the unit vector $\hat{\mathbf{g}}$ along the **shortest** axis.

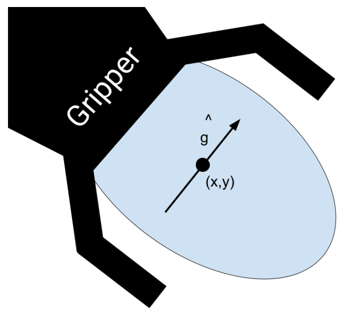

Figure 2: Grasp Visualization

Solution:
Because we are only working in 2 dimensions, the shortest axis of our ellipse estimation must be orthogonal to the major axis of our ellipse, per the definition of the minor axis. There are multiple possible ways that students can derive the particular vector $\hat{\mathbf{g}}$ that corresponds to this direction:

- Observing that the provided $\hat{\mathbf{m}}$ vector is the same as their result from the previous subpart, we can observe that the shortest axis must be the secondary right principal vector, or the second row of the matrix $\mathbf{V}^\top$ that we derived. As a normalized vector, this automatically gives us

$$\hat{\mathbf{g}} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

- Irrespective of the previous subpart, students can observe that the grasp axis should be a vector orthogonal to $\hat{\mathbf{m}}$, and then produce such a vector by inspection:

$$\langle \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 0$$

  Thus,

$$\hat{\mathbf{g}} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

- Alternatively to inspection, with an arbitrary identity basis vector (say $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$), we can extend an orthogonal basis with Gram-Schmidt to get

$$\mathbf{q} = \mathbf{e}_1 - \mathbf{proj}_{\hat{\mathbf{m}}} \mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

Normalizing, we see

$$\hat{\mathbf{g}} = \sqrt{2}\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix} = \begin{bmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{bmatrix}$$

All of the above gave us $\hat{\mathbf{g}} = \begin{bmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{bmatrix}$, but the vector $\hat{\mathbf{g}} = \begin{bmatrix}-\frac{1}{\sqrt{2}}\\ -\frac{1}{\sqrt{2}}\end{bmatrix}$ is also a valid solution.

Finally, we need to perform *motion planning*, so that we can actually move the gripper to the desired grasp position. For the sake of this problem, let us assume that the gripper's position can be modeled by the following discrete-time system:

$$\mathbf{x}[i+1] = \mathbf{B}\mathbf{x}[i] + \mathbf{c}u[i]$$

where $\mathbf{x}[i] = \begin{bmatrix}a\\ b\end{bmatrix}$ is the current position of our gripper at timestep i, $u[i]$ is our scalar control input at timestep i.

(d) (10 points) Assume that the gripper begins at state $\mathbf{x}[0] = \mathbf{0}$. Express the state of the gripper at timestep 4 as the result of a matrix-vector product of the form:

$$\mathbf{x}[4] = \mathbf{C}\mathbf{u}$$

where $\mathbf{u} = \begin{bmatrix}u[0] & u[1] & u[2] & u[3]\end{bmatrix}^\top$ is a vector of the first four control inputs, in terms of $\mathbf{B}$ and $\mathbf{c}$.

Solution:
To derive our desired matrix-vector expression for $\mathbf{x}[4]$, we can symbolically unroll the system equations for four timesteps as follows:

$$\mathbf{x}[1] = \mathbf{B}\mathbf{x}[0] + \mathbf{c}u[0] = \mathbf{c}u[0]$$
$$\mathbf{x}[2] = \mathbf{B}\mathbf{x}[1] + \mathbf{c}u[1] = \mathbf{B}(\mathbf{c}u[0]) + \mathbf{c}u[1] = \mathbf{B}\mathbf{c}u[0] + \mathbf{c}u[1]$$
$$\mathbf{x}[3] = \mathbf{B}\mathbf{x}[2] + \mathbf{c}u[2] = \mathbf{B}(\mathbf{B}\mathbf{c}u[0] + \mathbf{c}u[1]) + \mathbf{c}u[2] = \mathbf{B}^2\mathbf{c}u[0] + \mathbf{B}\mathbf{c}u[1] + \mathbf{c}u[2]$$
$$\mathbf{x}[4] = \mathbf{B}\mathbf{x}[3] + \mathbf{c}u[3] = \mathbf{B}(\mathbf{B}^2\mathbf{c}u[0] + \mathbf{B}\mathbf{c}u[1] + \mathbf{c}u[2]) + \mathbf{c}u[3] = \mathbf{B}^3\mathbf{c}u[0] + \mathbf{B}^2\mathbf{c}u[1] + \mathbf{B}\mathbf{c}u[2] + \mathbf{c}u[3]$$

Writing this in matrix-vector form, we can see that:

$$\mathbf{x}[4] = \mathbf{C}\mathbf{u}$$

where

$$\mathbf{C} = \begin{bmatrix}\mathbf{B}^3\mathbf{c} & \mathbf{B}^2\mathbf{c} & \mathbf{B}\mathbf{c} & \mathbf{c}\end{bmatrix}$$

(e) (10 points) Assume that the state of the gripper at timestep 4 can be modeled as:

$$\mathbf{x}[4] = \mathbf{C}\mathbf{u} = \begin{bmatrix}-2 & 1 & 2 & -1 \\ 2 & 1 & -2 & -1\end{bmatrix}\begin{bmatrix}u[0] \\ u[1] \\ u[2] \\ u[3]\end{bmatrix}$$

and that our desired grasp position is $\mathbf{x}^* = \begin{bmatrix}8 \\ 16\end{bmatrix}$. For our application, we need to reach this position in four timesteps, but would like to be as efficient as possible. In particular we would like to find the smallest vector of inputs $\mathbf{u}^*$, $\mathbf{u}$ *where* $\|\mathbf{u}\|$ *is minimized*, such that $\mathbf{x}^* = \mathbf{C}\mathbf{u}^*$ (we reach our desired position).

Given that the compact SVD of $\mathbf{C}$ is

$$\mathbf{C} = \begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}4 & 0 \\ 0 & 2\end{bmatrix}\begin{bmatrix}-\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}}\end{bmatrix}$$

Provide the vector $\mathbf{u}^*$.

Solution: Because $\mathbf{C}$ is a wide matrix with full row rank, we know that there will be infinitely many solutions. However, we can derive the minimum-norm solution, our desired $\mathbf{u}^*$ using the Moore-Penrose Pseudoinverse. As we have learned, there are two valid forms for the pseudoinverse, and we can use either of them for the purposes of this question:

- $\mathbf{C}^{\dagger} = \mathbf{V}\mathbf{\Sigma}^{\dagger}\mathbf{U}^{\top}$ We have that the SVD of $\mathbf{C}$ is

$$\mathbf{C} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

Thus, the pseudoinverse is

$$\mathbf{C}^{\dagger} = \mathbf{V}\mathbf{\Sigma}^{\dagger}\mathbf{U}^{\top} = \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 \\ 0 & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 \\ 0 & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{4\sqrt{2}} & -\frac{1}{4\sqrt{2}} \\ \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} \end{bmatrix} = \begin{bmatrix} -\frac{1}{8} & \frac{1}{8} \\ \frac{1}{4} & \frac{1}{4} \\ \frac{1}{8} & -\frac{1}{8} \\ -\frac{1}{4} & -\frac{1}{4} \end{bmatrix}$$

- Alternatively, $\mathbf{C}^{\dagger} = \mathbf{C}^{\top}(\mathbf{C}\mathbf{C}^{\top})^{-1} = \begin{bmatrix} -2 & 2 \\ 1 & 1 \\ 2 & -2 \\ -1 & -1 \end{bmatrix} \left(\begin{bmatrix} -2 & 1 & 2 & -1 \\ 2 & 1 & -2 & -1 \end{bmatrix} \begin{bmatrix} -2 & 2 \\ 1 & 1 \\ 2 & -2 \\ -1 & -1 \end{bmatrix} \right)^{-1}$

$$= \begin{bmatrix} -2 & 2 \\ 1 & 1 \\ 2 & -2 \\ -1 & -1 \end{bmatrix} \left(\begin{bmatrix} 10 & -6 \\ -6 & 10 \end{bmatrix} \right)^{-1} = \begin{bmatrix} -2 & 2 \\ 1 & 1 \\ 2 & -2 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} \frac{5}{32} & \frac{3}{32} \\ \frac{3}{32} & \frac{5}{32} \end{bmatrix} = \begin{bmatrix} -\frac{1}{8} & \frac{1}{8} \\ \frac{1}{4} & \frac{1}{4} \\ \frac{1}{8} & -\frac{1}{8} \\ -\frac{1}{4} & -\frac{1}{4} \end{bmatrix}$$

Once we have the pseudoinverse, the desired minimum set of control inputs can be computed using

$$\mathbf{u}^{*} = \mathbf{C}^{\dagger}\mathbf{x}^{*}$$

which gives us

$$\mathbf{u}^{*} = \begin{bmatrix} -\frac{1}{8} & \frac{1}{8} \\ \frac{1}{4} & \frac{1}{4} \\ \frac{1}{8} & -\frac{1}{8} \\ -\frac{1}{4} & -\frac{1}{4} \end{bmatrix} \begin{bmatrix} 8 \\ 16 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ -1 \\ -6 \end{bmatrix}$$

The optimal set up inputs to reach our goal is

- $u[0] = 1$
- $u[1] = 6$
- $u[2] = -1$
- $u[3] = -6$

MT3.4 (45 Points) Solving the Damped Harmonic Oscillator

In this problem, we will solve the damped harmonic oscillator, a classical problem in Physics. Suppose you have a spring, whose one side is attached to a wall and the other to a ball of mass m. Using Newton's 2nd law of motion, the force in a spring, and the viscocity of a fluid in which the spring is in, we can derive a governing differential equation of this system as:

$$\ddot{y}(t) = -\frac{c}{m}\dot{y}(t) - \frac{k}{m}y(t) \tag{1}$$

where $y(t)$ is the displacement of the mass, k is the spring constant and c is the viscous damping coefficient. For simplicitly, assume $m = 1$, so we get that:

$$\ddot{y}(t) = -c\dot{y}(t) - ky(t) \tag{2}$$

(a) (10 points) Set up the state space VDE to describe our system using $\mathbf{x}(t) = \begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix}$. That is, find the matrix $\mathbf{A}$ such that

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) \tag{3}$$

Solution:

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} \dot{y}(t) \\ \ddot{y}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k & -c \end{bmatrix} \begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix} = \mathbf{A}\mathbf{x}(t) \tag{4}$$

(b) (10 points) Suppose that $c = 2$, $k = 1$. Solve for the eigenvalues of the above system. Is the system stable?

Solution:

Compute the characteristic polynomial and find its roots.

$$\det(\lambda \mathbf{I} - \mathbf{A}) = \det\left(\begin{bmatrix} \lambda & -1 \\ 1 & \lambda + 2 \end{bmatrix}\right) = \lambda^2 + 2\lambda + 1 = 0 \tag{5}$$

which means that

$$\lambda = -1 \tag{6}$$

Since the only eigenvalue is negative. This implies that the system is stable.

(c) (15 points) Suppose that $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t)$ with $c = 26$ and $k = 25$, we get that $\mathbf{A}$ can be diagonalized as

$$\mathbf{A} = \begin{bmatrix} -1 & -1 \\ 25 & 1 \end{bmatrix} \begin{bmatrix} -25 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{24} & \frac{1}{24} \\ -\frac{25}{24} & -\frac{1}{24} \end{bmatrix} \tag{7}$$

Given that the system starts at the initial condition $y(0) = 1$ and $\dot{y}(0) = 0$, solve for $y(t)$. What is the final position of the mass as t approaches infinity? In other words, what is $\lim_{t \to \infty} y(t)$?

Hint: Recall the specific form in which we want the solution to be found in. How can you determine the final position of the mass using the solution?

Solution: We get that:

$$\mathbf{x}(t) = \begin{bmatrix} -1 & -1 \\ 25 & 1 \end{bmatrix} \begin{bmatrix} e^{-25t} & 0 \\ 0 & e^{-t} \end{bmatrix} \frac{1}{24} \begin{bmatrix} 1 & 1 \\ -25 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{8}$$

$$= \frac{1}{24} \begin{bmatrix} -1 & -1 \\ 25 & 1 \end{bmatrix} \begin{bmatrix} e^{-25t} & 0 \\ 0 & e^{-t} \end{bmatrix} \begin{bmatrix} 1 \\ -25 \end{bmatrix} \tag{9}$$

$$= \frac{1}{24} \begin{bmatrix} -1 & -1 \\ 25 & 1 \end{bmatrix} \begin{bmatrix} e^{-25t} \\ -25e^{-t} \end{bmatrix} \tag{10}$$

$$= \frac{1}{24} \begin{bmatrix} -e^{-25t} + 25e^{-t} \\ 25e^{-25t} - 25e^{-t} \end{bmatrix} \tag{11}$$

Finally, to determine the final position of the mass, take the limit of $y(t)$ as t goes to infinity. By taking this limit, we find the displacement to which the mass on the spring converges.

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty} \frac{1}{24}(-e^{-25t} + 25e^{-t}) \tag{12}$$

$$= 0 \tag{13}$$

which makes sense since the spring will eventually reach its relaxed state, so the final displacement of the mass is 0.

(d) (10 points) Now suppose that there is an external force that applies a constant force to the spring. In other words, the new governing equation is

$$\ddot{y}(t) = -c\dot{y}(t) - ky(t) + F \tag{14}$$

where F is some constant. Suppose that $y_h(t)$ is the homogeneous solution. Solve for the general solution of the system in terms of $y_h(t)$, F, c, and k. You may not need all of these.

Solution: New final position will be the homogeneous solution from above plus some function of the added particular solution in the previous part. Again, if student gets previous parts wrong, can still get full credit here if correctly solves particular.

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} \dot{y}(t) \\ \ddot{y}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k & -c \end{bmatrix} \begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix} + \begin{bmatrix} 0 \\ F \end{bmatrix} = \mathbf{A}\mathbf{x}(t) + \mathbf{b} \tag{15}$$

We already know from the previous part what the homogeneous solution is, so now it is just necessary to find the particular solution. Since the forced response is constant, we guess that the particular solution must also be a scalar. Thus,

$$\mathbf{0} = \mathbf{A}\mathbf{x}_p(t) + \mathbf{b} \tag{16}$$

$$\begin{bmatrix} 0 \\ -F \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k & -c \end{bmatrix} \begin{bmatrix} y_p(t) \\ \dot{y}_p(t) \end{bmatrix} \tag{17}$$

$$= \begin{bmatrix} 0 & 1 \\ -k & -c \end{bmatrix} \begin{bmatrix} y_p \\ 0 \end{bmatrix} \tag{18}$$

$$= \begin{bmatrix} 0 \\ -ky_p \end{bmatrix} \tag{19}$$

which clearly shows that $y_p = \frac{F}{k} = \frac{F}{k}$. Therefore, we have the following solution for $y(t)$:

$$y(t) = y_h(t) + \frac{F}{k} \tag{20}$$

- In some parts of a problem, we may ask you to establish a certain result—for example, ”show this” or ”prove that.” Even if you’re unable to establish the result that we ask of you, you may still take that result for granted—and use it in any subsequent part of the problem.
- If we ask you to provide a ”reasonably simple expression” for something, by default we expect your expression to be in closed form—one *not* involving a sum $\sum$ or an integral $\int$—*unless* we explicitly tell you otherwise.
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- **(10 Points) (Pledge of Academic Integrity)** Hand-copy, sign, and date the single-line text (which begins with *I have read, . . .*) of the Pledge of Academic Integrity on page 1 of this document. Your solutions will *not* be evaluated without this.
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- **This document consists of pages numbered 1 through 18.** Verify that your copy of the exam is free of anomalies, and contains all of the specified number of pages. If you find a defect in your copy, contact the teaching staff immediately.
- This exam is designed to be completed within 70 minutes. However, you may use up to 80 minutes total—*in one sitting*—to tackle the exam.

  The exam starts at 8:10 pm California time. Your allotted window begins with respect to this start time. Students who have official accommodations of $1.5\times$ and $2\times$ time windows have 120 and 160 minutes, respectively.
- **This exam is closed book.** You may not use or access, or cause to be used or accessed, any reference in print or electronic form at any time during the exam, except three double-sided 8.5" $\times$ 11" sheets of handwritten, original notes having no appendage.

  Collaboration is not permitted.

  Computing, communication, and other electronic devices (except dedicated time-keepers) must be turned off.

  Scratch paper will be provided to you; ask for more if you run out. You may not use your own scratch paper. Additionally, pages 15 and 16 can also be used as scratch paper, **but we will not grade these pages.**
- Please write neatly and legibly, because *if we can't read it, we can't evaluate it.*
- For each problem, limit your work to the space provided specifically for that problem. *No other work will be considered. For example, we will not evaluate scratch work. No exceptions.*
- Unless explicitly waived by the specific wording of a problem, you must explain your responses (and reasoning) succinctly, but clearly and convincingly.