FIRST Name ____________ LAST Name ____________ SID (All Digits):__________

- **(5 Points)** On *every* page, print legibly your name and ALL digits of your SID. For every page on which you do not write your name and SID, you forfeit a point, up to the maximum 5 points.

- **(10 Points) (Pledge of Academic Integrity)** Hand-copy, sign, and date the single-line text (which begins with *I have read, . . .*) of the Pledge of Academic Integrity on page 3 of this document. Your solutions will *not* be evaluated without this.

- **Urgent Contact with the Teaching Staff:** In case of an urgent matter, raise your hand if in-person, or send an email to `eecs16a@berkeley.edu` if online.

- **This document consists of pages numbered 1 through 14.** Verify that your copy of the exam is free of anomalies, and contains all of the specified number of pages. If you find a defect in your copy, contact the teaching staff immediately.

- This exam is designed to be completed within 70 minutes. However, you may use up to 80 minutes total—*in one sitting*—to tackle the exam.

  The exam starts at 8:10 pm California time. Your allotted window begins with respect to this start time. Students who have official accommodations of $1.5\times$ and $2\times$ time windows have 120 and 160 minutes, respectively.

- **This exam is closed book.** You may not use or access, or cause to be used or accessed, any reference in print or electronic form at any time during the exam, except one double-sided $8.5" \times 11"$ sheet of handwritten, original notes having no appendage.

  Collaboration is not permitted.

  Computing, communication, and other electronic devices (except dedicated timekeepers) must be turned off.

  Scratch paper will be provided to you; ask for more if you run out. You may not use your own scratch paper.

FIRST Name ____________ LAST Name ____________ SID (All Digits): ____________

- Please write neatly and legibly, because *if we can't read it, we can't evaluate it.*
- For each problem, limit your work to the space provided specifically for that problem. *No other work will be considered. For example, we will not evaluate scratch work. No exceptions.*
- Unless explicitly waived by the specific wording of a problem, you must explain your responses (and reasoning) succinctly, but clearly and convincingly.
- In some parts of a problem, we may ask you to establish a certain result—for example, "show this" or "prove that." Even if you're unable to establish the result that we ask of you, you may still take that result for granted—and use it in any subsequent part of the problem.
- If we ask you to provide a "reasonably simple expression" for something, by default we expect your expression to be in closed form—one *not* involving a sum $\sum$ or an integral $\int$—*unless* we explicitly tell you otherwise.
- Noncompliance with these or other instructions from the teaching staff—*including, for example, commencing work prematurely, or continuing it beyond the allocated time window*—is a serious violation of the Code of Student Conduct.

FIRST Name ____________ LAST Name ____________ SID (All Digits):__________

Pledge of Academic Integrity

By my honor, I affirm that

(1) this document—which I have produced for the evaluation of my performance—reflects my original, bona fide work, and that I have neither provided to, nor received from, anyone excessive or unreasonable assistance that produces unfair advantage for me or for any of my peers;

(2) as a member of the UC Berkeley community, I have acted with honesty, integrity, respect for others, and professional responsibility—and in a manner consistent with the letter and intent of the campus Code of Student Conduct;

(3) I have not violated—nor aided or abetted anyone else to violate—the instructions for this exam given by the course staff, including, but not limited to, those on the cover page of this document; and

(4) More generally, I have not committed any act that violates—nor aided or abetted anyone else to violate—UC Berkeley, state, or Federal regulations, during this exam.

(10 Points) In the space below, hand-write the following sentence, verbatim. Then write your name in legible letters, sign, include your full SID, and date before submitting your work:

I have read, I understand, and I commit to adhere to the letter and spirit of the pledge above.

____

____

Full Name: ________________________ Signature: ________________________

Date: ________________________ Student ID: ________________________

FIRST Name ____________ LAST Name ____________ SID (All Digits):____________

Potentially Useful Facts That You May Use Without the Need to Prove Them:

- **Inner Product:** For every $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{C}^n$, we define $\langle \boldsymbol{x}, \boldsymbol{y} \rangle \triangleq \boldsymbol{x}^\mathsf{T} \boldsymbol{y}^* = \sum_{k=1}^{n} x_k \, y_k^*$.

  The complex conjugation may be omitted for vectors in $\mathbb{R}^n$—that is, for every $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$, we define $\langle \boldsymbol{x}, \boldsymbol{y} \rangle \triangleq \boldsymbol{x}^\mathsf{T} \boldsymbol{y} = \sum_{k=1}^{n} x_k \, y_k$.

- **Cauchy-Schwarz Inequality:** For all elements x and y in a vector space $\mathcal{V}$,

$$\big|\langle x, y \rangle\big| \leq ||x|| \, ||y||.$$

- **Triangle Inequality:** For all elements x and y in a vector space $\mathcal{V}$,

$$||x + y|| \leq ||x|| + ||y||.$$

- **Geometric Sum Formula** For all integers M and N, where $M \leq N$,

$$\sum_{\ell=M}^{N} \alpha^\ell = \begin{cases} \dfrac{\alpha^{N+1} - \alpha^M}{\alpha - 1} & \text{if } \alpha \neq 1 \\ N - M + 1 & \text{if } \alpha = 1. \end{cases}$$

- **Angle Between Vectors:** The angle θ between two *nonzero* elements x and y in a *real* vector space satisfies

$$\theta = \arccos \frac{\langle x, y \rangle}{||x|| \, ||y||}, \qquad \text{and} \qquad \cos\theta = \frac{\langle x, y \rangle}{||x|| \, ||y||}.$$

  Whether in a real or complex vector space, if $\langle x, y \rangle = 0$, we say x and y are orthogonal, and we denote this by $x \perp y$.

- **Sum of the first n positive integers:** $$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}.$$

- **Sum of the squares of the first n positive integers:** $$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.$$

- **Polynomials**
  - Any *nonzero* polynomial $p(t) = \sum_{k=0}^{n} a_k t^k$ in a real variable t, having real coefficients a_k, of degree $n \geq 0$, has exactly n roots, inclusive of multiplicity (i.e., root repetition)—real or complex.
  - Any polynomial $p(t) = \sum_{k=0}^{n} a_k t^k$ of degree $n \geq 0$ is infinitely differentiable—that is, it has derivatives of all orders.

FIRST Name ______________ LAST Name ______________ SID (All Digits):____________

MT1.1 (35 Points) Consider a vector whose entries are the first n positive integers:

$$\boldsymbol{x} = \begin{bmatrix} 1 \\ 2 \\ \vdots \\ n \end{bmatrix}, \text{ where } n \in \{2, 3, 4, \ldots\}.$$

(a) (15 Points) Determine each of the following norms for $\boldsymbol{x}$. Each of your expressions must be in closed form—not left as a summation.

(i) (5 Points) The ℓ_∞-norm $||\boldsymbol{x}||_\infty$.

(ii) (5 Points) The ℓ_1-norm $||x||_1$.

(iii) (5 Points) The Euclidean (ℓ_2) norm $||\boldsymbol{x}||_2$.

MT1.1 (Continued)

(b) (20 Points) Consider the vectors $\boldsymbol{y}$ and $\boldsymbol{z}$ defined as

$$\boldsymbol{y} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad \text{and} \qquad \boldsymbol{z} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}.$$

For each of these two vectors, construct a corresponding *nonzero* orthogonal vector—in particular, construct $\boldsymbol{y}_\perp$ and $\boldsymbol{z}_\perp$ such that

$$\langle \boldsymbol{y}, \boldsymbol{y}_\perp \rangle = 0, \qquad \text{and} \qquad \langle \boldsymbol{z}, \boldsymbol{z}_\perp \rangle = 0.$$

There is an uncountably-infinite set of answers for each. Construct only one for each, and explain (briefly) your thought process.

FIRST Name ____________ LAST Name ____________ SID (All Digits):____________

MT1.2 (60 Points) Throughout this problem—except for part numbering—i denotes $\sqrt{-1}$, and the Cartesian form of a complex number z is $z = a + ib$, where $a, b \in \mathbb{R}$.

(a) (40 Points) Express each of the following expressions in a Cartesian form.

(i) (10 Points) $\frac{1}{i}$

(ii) (10 Points) $(1 + i)^2$

(iii) (10 Points) i^4

MT1.2 (Continued)

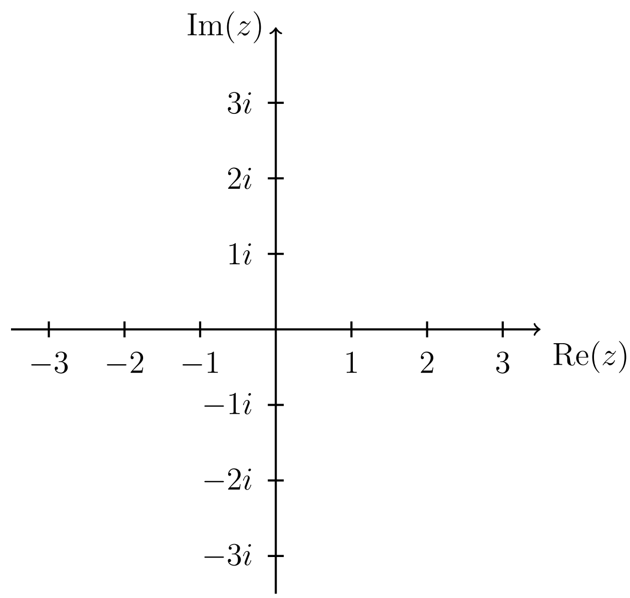
(iv) (10 Points) Show that $\sum_{k=0}^{3} i^k = 0$, and determine $\sum_{k=0}^{1000} i^k$ in Cartesian form.

MT1.2 (Continued)

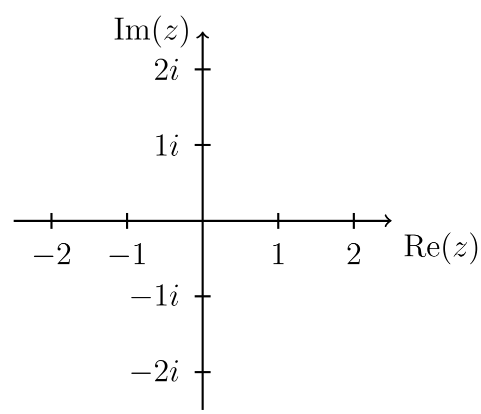
(b) (20 Points) For each of the following conditions, provide a well-labeled diagram of all points z in the complex plane that satisfy the stated condition.

Explain your work succinctly, and shade or otherwise highlight, the relevant region(s).

(i) (10 Points) $|2z - 4| \leq 2$.



(ii) (10 Points) $|z - i| \leq |z + i|$.



FIRST Name ____________ LAST Name ____________ SID (All Digits):__________

MT1.3 (35 Points) Consider the four vectors φ_k, where $k = 0, 1, 2, 3$, shown below:

$$\varphi_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \varphi_1 = \begin{bmatrix} 1 \\ i \\ -1 \\ -i \end{bmatrix}, \quad \varphi_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \quad \varphi_3 = \begin{bmatrix} 1 \\ -i \\ -1 \\ i \end{bmatrix}$$

(a) (5 Points) Show that φ_1 and φ_3 are orthogonal.

(b) (10 Points) Determine $||\varphi_k||^2$ for all $k = 0, 1, 2, 3$.

Hint: You may determine the norm for one of the four vectors, and then provide a brief, convincing reason why the other vectors have the same norm.

MT1.3 (Continued)

(c) (20 Points) It turns out that all the four vectors $\boldsymbol{\varphi}_k$, $k = 0, 1, 2, 3$ are mutually orthogonal—do *not* bother proving it here.

Assuming this mutual orthogonality, express the first canonical unit vector $\boldsymbol{e}_1$ in $\mathbb{R}^4$ as a linear combination of $\boldsymbol{\varphi}_k$, $k = 0, 1, 2, 3$. That is, determine the coefficients α_k in

$$\boldsymbol{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \sum_{k=0}^{3} \alpha_k \, \boldsymbol{\varphi}_k = \alpha_0 \boldsymbol{\varphi}_0 + \alpha_1 \boldsymbol{\varphi}_1 + \alpha_2 \boldsymbol{\varphi}_2 + \alpha_3 \boldsymbol{\varphi}_3.$$

MT1.4 (35 Points) Let $C^\infty(\mathbb{R})$ denote the vector space of functions of a continuous variable that are infinitely-differentiable over the entire real axis. A function belongs to this vector space if, and only if, it has derivatives of all orders—that is, all its derivatives are continuous. An element in this vector space is also called a C^∞ function (C^∞ is pronounced "C-infinity") or, easier yet, a *smooth* function.

Consider the finite set of smooth functions $\varphi_k : \mathbb{R} \to \mathbb{R}$, for $k = 0, 1, 2, \ldots, n$, where

$$\forall t \in \mathbb{R}, \qquad \varphi_0(t) = 1, \quad \varphi_1(t) = t, \quad \ldots \quad , \quad \varphi_k(t) = t^k, \quad \ldots \quad , \quad \varphi_n(t) = t^n$$

for some nonnegative integer n.

(a) (15 Points) Show that $\varphi_0, \ldots, \varphi_n$ are linearly independent.

Note: The zero element in $C^\infty(\mathbb{R})$ is the function that is zero for all $t \in \mathbb{R}$.

MT1.4 (Continued)

(b) (20 Points) Let $\mathcal{P}_n$ denote a real-valued vector space of polynomials of degree less than, or equal to, n, where n is a nonnegative integer and $t \in \mathbb{R}$. You can think of $\mathcal{P}_n$ as the set constructed from all possible real linear combinations of $\varphi_k(t)$ for $k = 0, 1, \ldots, n$. A generic polynomial in $\mathcal{P}_n$ can be expressed as follows:

$$p(t) = \sum_{k=0}^{n} p_k\, \varphi_k(t) = \sum_{k=0}^{n} p_k\, t^k = \underbrace{\begin{bmatrix} p_0 & p_1 & \cdots & p_n \end{bmatrix}}_{\boldsymbol{p}^\top} \underbrace{\begin{bmatrix} 1 \\ t \\ \vdots \\ t^n \end{bmatrix}}_{\boldsymbol{f}(t)} = \boldsymbol{p}^\top \boldsymbol{f}(t),$$

where $\boldsymbol{f}(t) \in \mathbb{R}^{n+1}$ denotes the vector of monomials (you can think of $\boldsymbol{f}$ as a vector-valued function of the continuous variable t), $\boldsymbol{p} \in \mathbb{R}^{n+1}$ denotes the vector of the coefficients, and $^\top$ denotes transpose.

Define $\mathcal{V} \subseteq \mathcal{P}_n$ as the subset of all polynomials in $\mathcal{P}_n$ that have $t = 0$ <u>and</u> $t = 1$ as roots. That is,

$$\mathcal{V} = \left\{ v(t) = \sum_{k=0}^{n} v_k\, t^k \;\middle|\; v(0) = 0, v(1) = 0, v_k \in \mathbb{R}, k = 0, \ldots, n \right\}.$$

Show that $\mathcal{V}$ is a subspace.

MT1.5 (20 Points) Let x and y denote two elements in a real vector space $\mathcal{V}$ defined over the scalar field $\mathbb{R}$. Assume that a norm $||\cdot|| : \mathcal{V} \to \mathbb{R}$ has been defined on $\mathcal{V}$ (though not necessarily induced by an inner product).

Show that

$$-||x-y|| \leq ||x|| - ||y|| \leq ||x-y||.$$

Note: You may *not* assume that the vector space $\mathcal{V}$ is of a specific type (such as $\mathbb{R}^n$). You may *not* assume that the norm is of a specific type (such as the ℓ_2-norm). And you may *not* assume that an inner product has been defined on $\mathcal{V}$.

To receive full credit, you must resort only to the properties of a generic vector space and a valid, but otherwise unspecified, norm.

However, if you use a specific vector space (such as $\mathbb{R}^n$) in your reasoning; if you can't show the result without appeal to a specific norm; or if you assume a properly-defined inner product to argue your way to the result, you may receive a credit of at most 80% for this problem, provided your work contains no other shortcoming.

Hint: A carrier pigeon has told you that $x = (x-y)+y$—and, similarly, that $y = (y-x)+x$.