EECS 16A: Foundations of Signals, Dynamical Systems, and Information Processing Exam 1

Department of Electrical Engineering and Computer Sciences
University of California, Berkeley 16 September 2025

FIRST Name: Ortho LAST Name: Gonna SID (All Digits): 0123456789

- **(5 Points)** On *every* page, print legibly your name and ALL digits of your SID. For every page on which you do not write your name and SID, you forfeit a point, up to the maximum 5 points.
- **(10 Points) (Pledge of Academic Integrity)** Hand-copy, sign, and date the single-line text (which begins with *I have read, ...*) of the Pledge of Academic Integrity on page 3 of this document. Your solutions will *not* be evaluated without this.
- **Urgent Contact with the Teaching Staff:** In case of an urgent matter, raise your hand.
- **This document consists of pages numbered 1 through 12.** Verify that your copy of the exam is free of anomalies, and contains all of the specified number of pages. If you find a defect in your copy, contact the teaching staff immediately.
- This exam is designed to be completed within 70 minutes. However, you may use up to 80 minutes total—*in one sitting*—to tackle the exam.

  The exam starts at 8:10 pm California time. Your allotted window begins with respect to this start time. Students who have official accommodations of 1.5× and 2× time windows have 120 and 160 minutes, respectively.
- **This exam is closed book.** You may not use or access, or cause to be used or accessed, any reference in print or electronic form at any time during the exam, except one double-sided 8.5" × 11" sheet of handwritten, original notes having no appendage.

  Collaboration is not permitted.

  Computing, communication, and other electronic devices (except dedicated timekeepers) must be turned off.

  Scratch paper will be provided to you; ask for more if you run out. You may not use your own scratch paper.
- Please write neatly and legibly, because *if we can't read it, we can't evaluate it.*
- For each problem, limit your work to the space provided specifically for that problem. *No other work will be considered. For example, we will not evaluate scratch work. No exceptions.*
- Unless explicitly waived by the specific wording of a problem, you must explain your responses (and reasoning) succinctly, but clearly and convincingly.
- In some parts of a problem, we may ask you to establish a certain result—for example, "show this" or "prove that." Even if you're unable to establish the result that we ask of you, you may still take that result for granted—and use it in any subsequent part of the problem.

- If we ask you to provide a "reasonably simple expression" for something, by default we expect your expression to be in closed form—one *not* involving a sum $\sum$ or an integral $\int$—*unless* we explicitly tell you otherwise.
- Noncompliance with these or other instructions from the teaching staff—*including, for example, commencing work prematurely, or continuing it beyond the allocated time window*—is a serious violation of the Code of Student Conduct.

FIRST Name: Ortho LAST Name: Gonna SID (All Digits): 0123456789

Pledge of Academic Integrity

By my honor, I affirm that

(1) this document—which I have produced for the evaluation of my performance—reflects my original, bona fide work, and that I have neither provided to, nor received from, anyone excessive or unreasonable assistance that produces unfair advantage for me or for any of my peers;

(2) as a member of the UC Berkeley community, I have acted with honesty, integrity, respect for others, and professional responsibility—and in a manner consistent with the letter and intent of the campus Code of Student Conduct;

(3) I have not violated—nor aided or abetted anyone else to violate—the instructions for this exam given by the course staff, including, but not limited to, those on the cover page of this document; and

(4) More generally, I have not committed any act that violates—nor aided or abetted anyone else to violate—UC Berkeley, state, or Federal regulations, during this exam.

(10 Points) In the space below, hand-write the following sentence, verbatim. Then write your name in legible letters, sign, include your full SID, and date before submitting your work:

I have read, I understand, and I commit to adhere to the letter and spirit of the pledge above.

I have read, I understand, and I commit to adhere to the letter and the spirit of the pledge above.

Full Name: Ortho Ganna

Signature: $\int_{-\infty}^{\infty} \psi_k(t)\psi_l(t)\,dt$

Date: 15 Sep 2025

Student ID: 0123456789

FIRST Name: Ortho LAST Name: Gonna SID (All Digits): 0123456789

E1.1 (45 Points) Linear Independence or Dependence of Functions

Consider a set $\mathsf{S} = \{\psi_0, \psi_1, \psi_2, \psi_3\}$ of real continuous-time signals where for all $t \in \mathbb{R}$,

$$\psi_0(t) = 1 \qquad\qquad \psi_1(t) = \sin^2\left(\frac{\pi}{2}t\right)$$

$$\psi_2(t) = \cos^2\left(\frac{\pi}{2}t\right) \qquad\qquad \psi_3(t) = \sin(\pi t)\,.$$

Note: You do *not* need *any* half- or double-angle trigonometric formula for *any* part of this problem. However, if you insist, here you go:

$$\sin^2\left(\frac{\alpha}{2}\right) = \frac{1-\cos\alpha}{2} \qquad \text{and} \qquad \cos^2\left(\frac{\alpha}{2}\right) = \frac{1+\cos\alpha}{2}.$$

(a) (15 Points) Show that the set of signals $\mathsf{A} = \{\psi_1, \psi_2\}$ is linearly independent. Provide a succinct yet clear and convincing explanation.

Assume coefficients α_1 and α_2 exist such that $\alpha_1\psi_1 + \alpha_2\psi_2 = 0$.
Note ~~this~~ the meaning of this is that $\alpha_1\psi_1(t) + \alpha_2\psi_2(t) = 0 \quad \forall t \in \mathbb{R}$.
Through a judicious choice of t, we now show that α_1 and α_2 must both be zero, thereby establishing the linear independence of $\{\psi_1, \psi_2\}$.

Let $t=0 \Rightarrow \alpha_1 \sin^2\left(\frac{\pi}{2}\cdot 0\right) + \alpha_2\cos^2\left(\frac{\pi}{2}\cdot 0\right) = \alpha_2 = 0$

Let $t=1 \Rightarrow \alpha_1 \underbrace{\sin^2\left(\frac{\pi}{2}\right)}_{1} + \alpha_2 \underbrace{\cos^2\left(\frac{\pi}{2}\right)}_{0} = 0 \Rightarrow \alpha_1 = 0$

Done!

(b) (15 Points) Determine whether the set of signals $\mathsf{B} = \{\psi_0, \psi_1, \psi_2\}$ is linearly independent. Provide a succinct yet clear and convincing explanation.

The set $B = \{\psi_0, \psi_1, \psi_2\}$ is linear dependent because $\psi_0 = \psi_1 + \psi_2$. Why? Because of the trigonometric identity $\sin^2\theta + \cos^2\theta = 1$. In our case,

$$\underbrace{\sin^2\left(\frac{\pi}{2}t\right)}_{\psi_1(t)} + \underbrace{\cos^2\left(\frac{\pi}{2}t\right)}_{\psi_2(t)} = \underbrace{1}_{\psi_0(t)} \quad \forall t \in \mathbb{R}.$$

So $\underbrace{1}_{\alpha_0 = 1}\cdot\psi_0 \underbrace{-1}_{\alpha_1=-1}\cdot\psi_1 \underbrace{-1}_{\alpha_2=-1}\psi_2 = 0$

which implies linear dependence.

FIRST Name: Ortho LAST Name: Gonna SID (All Digits): 0123456789

E1.1 (Continued)

(c) (15 Points) Consider the following inner product defined on $\mathsf{span}\,(\mathsf{S})$:

$$\langle \psi_k, \psi_\ell \rangle \triangleq \int_{-1}^{+1} \psi_k(t)\,\psi_\ell(t)\,dt.$$

Recall that if the angle between signals ψ_k and ψ_ℓ is denoted by $\theta_{k\ell}$, then

$$\cos\theta_{k\ell} = \frac{\langle \psi_k, \psi_\ell \rangle}{||\psi_k||\,||\psi_\ell||}.$$

Determine the angle θ_{13} between ψ_1 and ψ_3, and explain whether the set of signals $\mathsf{C} = \{\psi_1, \psi_3\}$ is linearly independent.

$$\langle \psi_1, \psi_3 \rangle = \int_{-1}^{1} \psi_1(t)\psi_3(t)\,dt = \int_{-1}^{1} \sin^2\left(\frac{\pi}{2}t\right)\sin(\pi t)\,dt.$$

The integration is over a symmetric interval around 0.
The integrand $\psi_1(t)\psi_3(t) = \sin^2\left(\frac{\pi}{2}t\right)\sin(\pi t)$ is the product of an even function ψ_1 and an odd function ψ_3. The integrand is thus odd in t.

Therefore the integral evaluates to zero, which implies

$$\langle \psi_1, \psi_3 \rangle = \int_{-1}^{1} \sin^2\left(\frac{\pi}{2}t\right)\sin(\pi t)\,dt = 0 \Rightarrow \langle \psi_1, \psi_3 \rangle = 0$$

$$\Rightarrow \psi_1 \perp \psi_3 \Rightarrow C = \{\psi_1, \psi_3\} \text{ is linearly independent.}$$

This was not required of you to show, but to see why the orthagonality of ψ_1 and ψ_3 implies their linear independence, let $\alpha_1\psi_1 + \alpha_3\psi_3 = 0$. Now take the inner product of both sides with ψ_1: $\alpha_1\langle\psi_1,\psi_1\rangle + \alpha_3\overbrace{\langle\psi_3,\psi_1\rangle}^{0} = \langle 0, \psi_1\rangle = 0 \Rightarrow \alpha_1 = 0$.

Now take the inner product with ψ_3 to obtain $\alpha_3 = 0$.

FIRST Name: Ortho LAST Name: Gonna SID (All Digits): 0123456789

E1.2 (25 Points) Parallelograms and Pythagoras

Consider the complex Euclidean vector space $\mathcal{V} = \mathbb{C}^n$. Show that the following identity holds:

$$\|\boldsymbol{x} + \boldsymbol{y}\|^2 + \|\boldsymbol{x} - \boldsymbol{y}\|^2 = 2\Big(\|\boldsymbol{x}\|^2 + \|\boldsymbol{y}\|^2\Big),$$

where $\|\cdot\|$ denotes the ℓ_2-norm (Euclidean norm).

If you're unable to show this for $\mathbb{C}^n$, you may receive partial credit if you show it for $\mathbb{R}^n$. In fact, the identity holds for every inner product space where the norm is induced by an inner product—that is, any inner product space in which for every element x the following holds: $\langle x, x\rangle = \|x\|^2$.

$$\|x+y\|^2 = \langle x+y, x+y\rangle = \langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle + \langle y,y\rangle$$

$$\|x-y\|^2 = \langle x-y, x-y\rangle = \langle x,x\rangle - \langle x,y\rangle - \langle y,x\rangle + \langle y,y\rangle$$

$$\|x+y\|^2 + \|x-y\|^2 = 2\langle x,x\rangle + \cancel{\langle x,y\rangle} + \cancel{\langle y,x\rangle} - \cancel{\langle x,y\rangle} - \cancel{\langle y,x\rangle} + 2\langle y,y\rangle$$

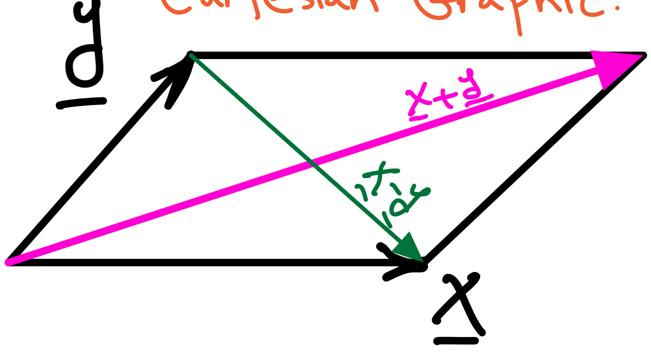
$$= 2\langle x,x\rangle + 2\langle y,y\rangle$$

$$= 2\|x\|^2 + 2\|y\|^2 \quad \Rightarrow$$

$$\boxed{\|x+y\|^2 + \|x-y\|^2 = 2\left(\|x\|^2 + \|y\|^2\right)}$$

Parallelogram Law

Cartesian Graphic:



Sum of the squared norms of the diagonals is twice the sum of the squared norms of the sides.

If $x \perp y$, the Parallelogram Law reduces to Pythagoras's Theorem.

FIRST Name: Ortho LAST Name: Gonna SID (All Digits): 0123456789

E1.3 (55 Points) Signal Families

A real continuous-time signal $\psi_0 : [0,1) \to \mathbb{R}$ is given by

$$\psi_0(t) = \begin{cases} 1 & 0 \le t < 1 \\ 0 & \text{elsewhere.} \end{cases}$$

Construct a family of signals ψ_n as follows:

$$\forall t \in \mathbb{R}, \qquad \psi_n(t) = \sqrt{2^n}\,\psi_0\left(2^n t\right), \qquad n = 0, 1, 2, 3, \ldots.$$

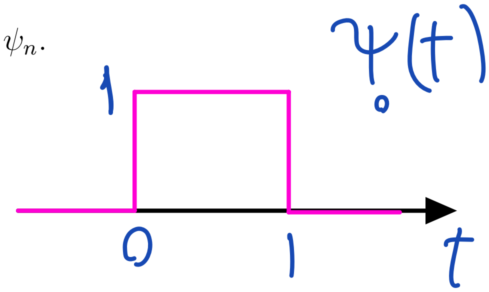
Hint: In one or more parts below, it may or may not be useful for you to know that $\psi_n(t)$ can be written as

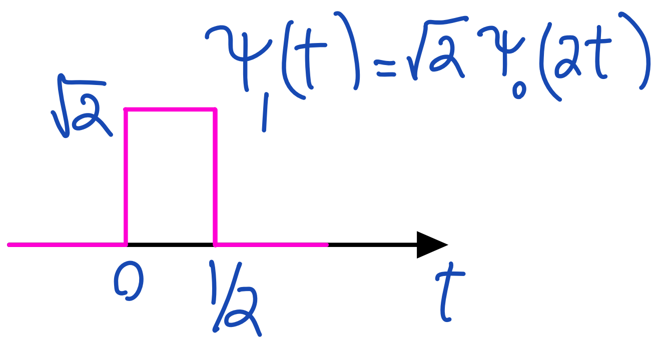
$$\psi_n(t) = \begin{cases} \sqrt{2^n} & 0 \le t < \dfrac{1}{2^n} \\ 0 & \text{elsewhere.} \end{cases}$$

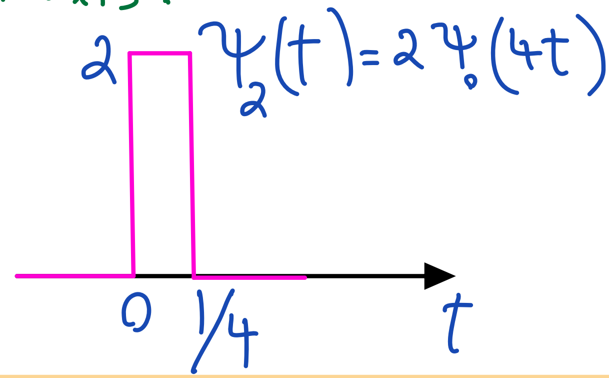
(a) (15 Points) Provide well-labeled plots of ψ_1, ψ_2, and ψ_n.

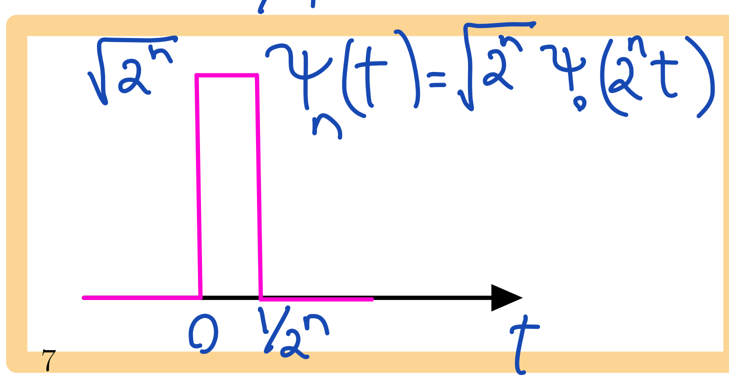
$$\psi_n(t) = \underbrace{\sqrt{2^n}}_{\text{Amplitude Scaling by } \sqrt{2^n}}\ \psi_0(\underbrace{2^n t}_{})$$

Contraction by a factor of 2^n along the time axis.









FIRST Name: Ortho LAST Name: Gonna SID (All Digits): 0123456789

E1.3 (Continued)

(b) (25 Points) Determine a simple expression (in terms of n) for each of the ℓ_1-norm, the ℓ_2-norm, and the ℓ_∞-norm of the n^{th} signal ψ_n. Provide succinct, yet clear and convincing work in the space allocated below for each norm.

(i) (10 Points) $\|\psi_n\|_1$

$$\|\psi_n\|_1 = \int_{-\infty}^{\infty} |\psi_n(t)|\,dt = \int_0^{1/2^n} \sqrt{2^n}\,dt = \frac{\sqrt{2^n}}{2^n} = \frac{1}{\sqrt{2^n}} = 2^{-n/2}$$

$$\Rightarrow \boxed{\|\psi_n\|_1 = \frac{1}{\sqrt{2^n}} = 2^{-n/2}}$$

(ii) (10 Points) $\|\psi_n\|_2$

$$\|\psi_n\|_2^2 = \int_{-\infty}^{\infty} |\psi_n(t)|^2\,dt = \int_0^{1/2^n} 2^n\,dt = \frac{2^n}{2^n} = 1 \Rightarrow$$

$$\boxed{\|\psi_n\|_2 = 1}$$

(iii) (5 Points) $\|\psi_n\|_\infty$

$$\boxed{\|\psi_n\|_\infty = \sup_t |\psi_n(t)| = \sqrt{2^n}}$$

In this case, the maximum and the least upper bound (i.e., supremum) are the same.

FIRST Name: Ortho LAST Name: Gonna SID (All Digits): 0123456789

E1.3 (Continued)

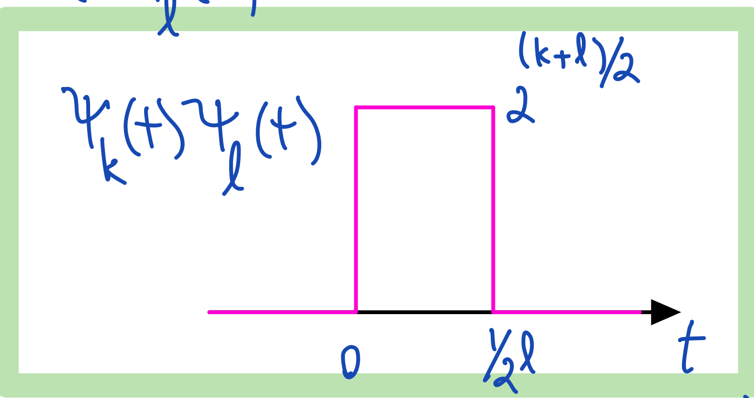
(c) (15 Points) Determine a reasonably simple expression for

$$\langle \psi_k, \psi_\ell \rangle \triangleq \int_{-\infty}^{\infty} \psi_k(t)\, \psi_\ell(t)\, dt,$$

where $k, \ell \in \{0, 1, 2, 3, \ldots\}$. Without loss of generality, you may assume that $k \leq \ell$.

$$\psi_k(t) = \begin{cases} 2^{k/2} & 0 \leqslant t \leqslant \frac{1}{2^k} \\ 0 & \text{elsewhere} \end{cases} \qquad \psi_\ell(t) = \begin{cases} 2^{\ell/2} & 0 \leqslant t \leqslant \frac{1}{2^\ell} \\ 0 & \text{elsewhere.} \end{cases}$$

If $k \leqslant \ell$, then $\frac{1}{2^\ell} \leqslant \frac{1}{2^k}$, so the product of $\psi_k(t)$ and $\psi_\ell(t)$ is nonzero for $t \in [0, \frac{1}{2^\ell})$.



$$\langle \psi_k, \psi_\ell \rangle = \int_{-\infty}^{\infty} \psi_k(t)\psi_\ell(t)\, dt = \int_0^{1/2^\ell} 2^{(k+\ell)/2}\, dt = \frac{2^{\frac{k+\ell}{2}}}{2^\ell}$$

$$\boxed{\langle \psi_k, \psi_\ell \rangle = 2^{(k-\ell)/2} = \sqrt{2^{k-\ell}}}$$

FIRST Name: Ortho LAST Name: Gonna SID (All Digits): 0123456789

E1.4 (60 Points) Subspaces

For each of the sets described below, determine whether it is a subspace. For each set, if you claim it is a subspace, you must prove it. If you claim it is not a subspace, you must show that it violates a necessary axiom.

(a) (12 Points) $\mathsf{A} = \left\{ \boldsymbol{x} \in \mathbb{R}^n \,\middle|\, \mu_{\boldsymbol{x}} \triangleq \frac{1}{n}\sum_{k=1}^{n} x_k = 0 \right\}$, where $\mu_{\boldsymbol{x}}$ denotes the average value (arithmetic mean) of the components of the vector $\boldsymbol{x}$.

Recognize $\mu_{\underline{x}} = \frac{1}{n} \mathbb{1}^T \underline{x} = \frac{1}{n} \langle \mathbb{1}, \underline{x} \rangle$, where $\mathbb{1}$ denotes the all-ones vector in $\mathbb{R}^n$. This means that all zero-mean vectors are orthogonal to the all-ones vector.

This is a subspace because (I) $\underline{0} \in A$; (II) If $\underline{x} \in A$, then $\mu_{\alpha\underline{x}} = \frac{1}{n}\langle \mathbb{1}, \alpha \underline{x}\rangle = \frac{\alpha}{n}\langle \mathbb{1}, \underline{x}\rangle = 0 \Rightarrow \alpha \underline{x} \in A$; and

(III) If $\underline{x}, \underline{y} \in A$, then $\mu_{\underline{x}+\underline{y}} = \frac{1}{n}\langle \mathbb{1}, \underline{x}+\underline{y}\rangle = \frac{1}{n}\left(\cancelto{0}{\langle \mathbb{1}, \underline{x}\rangle} + \cancelto{0}{\langle \mathbb{1}, \underline{y}\rangle}\right) = 0 \Longrightarrow$ $\underline{x}+\underline{y} \in A$.

A is a subspace

(b) (12 Points) $\mathsf{B} = \{x : \mathbb{R} \to \mathbb{C} \,|\, x(t) = 0 \text{ for all } |t| > 1\}$.

(I) The function $x : \mathbb{R} \to \mathbb{C}$, $x(t) = 0 \;\forall t \in \mathbb{R}$ belongs to B.

(II) If $x \in B$, then $y = \alpha x$ also satisfies $y(t) = \alpha x(t) = 0$ for $|t| > 1$, so $y \in B$.

(III) If $x, y \in B$, then $z = x + y$ also belongs to B, because $z(t) = x(t) + y(t) = 0$ for all $|t| > 1$, since each term in $x(t) + y(t)$ is zero for all $|t| > 1$.

B is a subspace

FIRST Name: Ortho LAST Name: Gonna SID (All Digits): 0123456789

E1.4 (Continued)

(c) (12 Points) The set of all discrete-time signals described below:

$$\mathsf{C} = \left\{ x : \mathbb{Z} \to \mathbb{C} \,\middle|\, x[0] = 1 \right\}.$$

C is not a subspace for any of the following reasons:

(I) The zero signal z, where $z[n]=0 \;\; \forall n \in \mathbb{Z}$ doesn't belong to C.

(II) If $x \in C$, then $y=\alpha x$ is such that $y[0]=\alpha x[0]=\alpha$. Clearly, for any scalar $\alpha \neq 1$, y won't belong to C. Closure under scalar multiplication fails.

(III) If $x, y \in C$, and $z=x+y$, then $z[0]=x[0]+y[0]=2 \Rightarrow z \notin C$

Closure under vector addition fails. You only needed to give one reason

(d) (12 Points) The set of all solutions of a first-order homogeneous differential equation:

$$\mathsf{D} = \left\{ y : \mathbb{R} \to \mathbb{C} \,\middle|\, \frac{dy(t)}{dt} + \alpha\, y(t) = 0 \text{ for some coefficient } \alpha \in \mathbb{C} \right\}.$$

Note: To tackle this question, you need *not* know how to solve differential equations. You need only know basic properties of derivatives.

D is a subspace.

(I) The zero function z such that $z(t)=0$ satisfies the differential equation.

(II) Suppose $y \in D$. Define $q=\alpha y$ for an arbitrary scalar α. Then

$$\frac{dq(t)}{dt} + \alpha\, q(t) = \alpha \frac{dy(t)}{dt} + \alpha y(t) = \alpha \left(\frac{dy(t)}{dt} + y(t) \right) = \alpha \cdot 0 = 0.$$

So, D is closed under scalar multiplication.

(III) Suppose $y, r \in D$, and let $v=y+r$. Then

$$\frac{dv(t)}{dt} + \alpha\, v(t) = \frac{d}{dt}\left[y(t)+r(t)\right] + \alpha\left[y(t)+r(t)\right] = \cancelto{0}{\frac{dy(t)}{dt} + \alpha y(t)} + \cancelto{0}{\frac{dr(t)}{dt} + \alpha r(t)} = 0$$

So, D is closed under vector addition.

FIRST Name: Ortho LAST Name: Gonna SID (All Digits): 0123456789

E1.4 (Continued)

(e) (12 points) The set of all bounded discrete-time signals $x : \mathbb{Z} \to \mathbb{C}$:

$$\ell_\infty(\mathbb{Z}) = \left\{ x : \mathbb{Z} \to \mathbb{C} \,\middle|\, \|x\|_\infty < \infty \right\}.$$

A discrete-time signal x is called bounded if there exists a nonnegative finite number B_x such that $|x[n]| \leq B_x$ for all integers n. In other words, x is called bounded if $\|x\|_\infty \leq B_x < \infty$ for some B_x, where $\|x\|$ denotes the ∞-norm of x defined by the Least Upper Bound of the set

$$\left\{ |x[n]| \,\middle|\, n \in \mathbb{Z} \right\}.$$

$\boxed{\ell_\infty(\mathbb{Z}) \text{ is a subspace.}}$

(I) The zero signal $z : \mathbb{Z} \to \mathbb{C}$ such that $z[n] = 0 \ \forall n \in \mathbb{Z}$ has $\|z\|_\infty = 0 < \infty \implies z \in \ell_\infty(\mathbb{Z})$

(II) Let $x \in \ell_\infty(\mathbb{Z})$ and define $y = \alpha x$ for an arbitrary scalar α. Then

$$\|y\|_\infty = \|\alpha x\|_\infty = \underbrace{|\alpha|}_{<\infty} \underbrace{\|x\|_\infty}_{<\infty} < \infty \implies y \in \ell_\infty(\mathbb{Z})$$

So, $\ell_\infty(\mathbb{Z})$ is closed under scalar multiplication.

(III) Let $x, y \in \ell_\infty(\mathbb{Z})$ and define $v = x + y$. By the Triangle Inequality, we have

$$\|v\|_\infty = \|x+y\|_\infty \leq \underbrace{\|x\|_\infty}_{<\infty} + \underbrace{\|y\|_\infty}_{<\infty} < \infty \implies v \in \ell_\infty(\mathbb{Z})$$

So, $\ell_\infty(\mathbb{Z})$ is closed under vector addition.