FIRST Name: Igen LAST Name: Viktor SID (All Digits): 0123456789

- **(5 Points)** On *every* page, print legibly your name and ALL digits of your SID. For every page on which you do not write your name and SID, you forfeit a point, up to the maximum 5 points.
- **(10 Points) (Pledge of Academic Integrity)** Hand-copy, sign, and date the single-line text (which begins with *I have read, . . .*) of the Pledge of Academic Integrity on page 3 of this document. Your solutions will *not* be evaluated without this.
- **Urgent Contact with the Teaching Staff:** In case of an urgent matter, raise your hand.
- **This document consists of pages numbered 1 through 15.** Verify that your copy of the exam is free of anomalies, and contains all of the specified number of pages. If you find a defect in your copy, contact the teaching staff immediately.
- This exam is designed to be completed within 70 minutes. However, you may use up to 80 minutes total—*in one sitting*—to tackle the exam.

  The exam starts at 8:10 pm California time. Your allotted window begins with respect to this start time. Students who have official accommodations of 1.5× and 2× time windows have 120 and 160 minutes, respectively.
- **This exam is closed book.** You may not use or access, or cause to be used or accessed, any reference in print or electronic form at any time during the exam, except two double-sided 8.5” × 11” sheets of handwritten, original notes having no appendage.

  Collaboration is not permitted.

  Computing, communication, and other electronic devices (except dedicated timekeepers) must be turned off.

  Scratch paper will be provided to you; ask for more if you run out. You may not use your own scratch paper.
- Please write neatly and legibly, because *if we can’t read it, we can’t evaluate it.*
- For each problem, limit your work to the space provided specifically for that problem. *No other work will be considered. For example, we will not evaluate scratch work. No exceptions.*
- Unless explicitly waived by the specific wording of a problem, you must explain your responses (and reasoning) succinctly, but clearly and convincingly.
- In some parts of a problem, we may ask you to establish a certain result—for example, ”show this” or ”prove that.” Even if you’re unable to establish the result that we ask of you, you may still take that result for granted—and use it in any subsequent part of the problem.

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- If we ask you to provide a "reasonably simple expression" for something, by default we expect your expression to be in closed form—one *not* involving a sum $\sum$ or an integral $\int$—*unless* we explicitly tell you otherwise.
- Noncompliance with these or other instructions from the teaching staff—*including, for example, commencing work prematurely, or continuing it beyond the allocated time window*—is a serious violation of the Code of Student Conduct.

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Pledge of Academic Integrity

By my honor, I affirm that

(1) this document—which I have produced for the evaluation of my performance—reflects my original, bona fide work, and that I have neither provided to, nor received from, anyone excessive or unreasonable assistance that produces unfair advantage for me or for any of my peers;

(2) as a member of the UC Berkeley community, I have acted with honesty, integrity, respect for others, and professional responsibility—and in a manner consistent with the letter and intent of the campus Code of Student Conduct;

(3) I have not violated—nor aided or abetted anyone else to violate—the instructions for this exam given by the course staff, including, but not limited to, those on the cover page of this document; and

(4) More generally, I have not committed any act that violates—nor aided or abetted anyone else to violate—UC Berkeley, state, or Federal regulations, during this exam.

(10 Points) In the space below, hand-write the following sentence, verbatim. Then write your name in legible letters, sign, include your full SID, and date before submitting your work:

I have read, I understand, and I commit to adhere to the letter and spirit of the pledge above.

I have read, I understand, and I commit to adhere to the letter and spirit of the pledge above.

Full Name: Igen Viktor Signature: IV

Date: 30 oct 2025 Student ID: 0123456789

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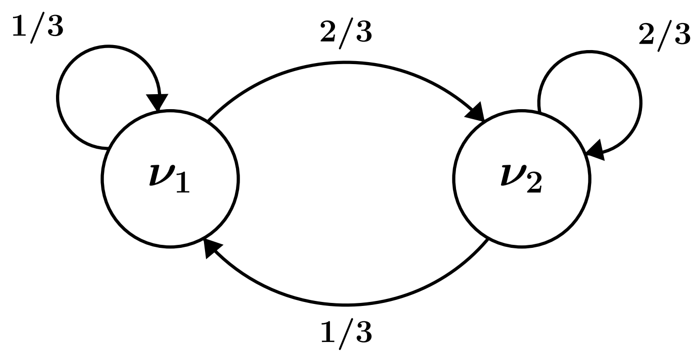
E2.1 (35 Points) Eigenanalysis of PageRank

Consider the two-node directed graph shown in the figure below:



The connectivity matrix for this graph is

$$\mathbf{C} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{2}{3} \end{bmatrix}.$$

This graph does *not* have equitable edge weights. For example, the two edges issued by Node ν_1 carry unequal weights. So do the two edges issued by Node ν_2.

In other words, the nodes do not fractionate their total unit quotas of "votes" equally between their respective targets. Node ν_1 is generous, preferring Node ν_2, whereas Node ν_2 is greedy, preferring itself. Nevertheless, the sum of the outgoing weights from each node is 1.

A state-space model of how the users of this network migrate from node to node is

$$\boldsymbol{s}[n+1] = \mathbf{C}\,\boldsymbol{s}[n],$$

where $\boldsymbol{s}[n] = \begin{bmatrix} s_1[n] \\ s_2[n] \end{bmatrix}$ denotes the state vector at time instance n. The state vector conveys the fraction of the population at each of the two nodes at time n. It is nonnegative (i.e., $\boldsymbol{s}[n] \succcurlyeq \mathbf{0}$) and is total-sum normalized (i.e., $\mathbf{1}^\mathsf{T}\boldsymbol{s}[n] = 1$) for all $n = 0, 1, 2, \ldots$.

The network is conservative—nobody enters, nobody leaves.

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(a) (20 Points) Determine every eigenvalue-eigenvector pair $(\lambda, \boldsymbol{v})$ of the matrix $\mathbf{C}$.

Method 1:

C is column-stochastic $\Rightarrow \lambda_1 = 1$. $\lambda_1 + \lambda_2 = tr(C) \Rightarrow \lambda_2 = 0$

$$\lambda_1 I - C = \begin{bmatrix} 1-1/3 & -1/3 \\ -2/3 & 1-2/3 \end{bmatrix} = \begin{bmatrix} 2/3 & -1/3 \\ -2/3 & 1/3 \end{bmatrix} \Rightarrow \underline{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\lambda_2 I - C = -C = \begin{bmatrix} -1/3 & -1/3 \\ -2/3 & -2/3 \end{bmatrix} \Rightarrow \underline{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\boxed{\left(\lambda_1 = 1, \underline{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)}$$

$$\boxed{\left(\lambda_2 = 0, \underline{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)}$$

Method 2: C is singular, with its columns identical. So it has an eigval $\lambda_2 = 0$, with corresp. eigvec $\underline{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. $tr(C) = \lambda_1 + \lambda_2 = 1 \Rightarrow \lambda_1 = 1$

Method 3: Long way to compute the eigenvalues:

$$|\lambda I - C| = \begin{vmatrix} \lambda - 1/3 & -1/3 \\ -2/3 & \lambda - 2/3 \end{vmatrix} = (\lambda - 1/3)(\lambda - 2/3) - \frac{2}{9} = \lambda^2 - \lambda = \lambda(\lambda - 1) = 0 \Rightarrow \begin{cases} \lambda_1 = 1 \\ \lambda_2 = 0 \end{cases}$$

(b) (5 Points) Determine the Google PageRank score vector $\boldsymbol{s}^*$ for the graph.

s^* is the total-sum-normalized version of $\underline{v}_1$:

$$s^* = \frac{1}{\mathbb{1}^T \underline{v}_1} \underline{v}_1 = \frac{1}{3} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \Rightarrow$$

$$\boxed{s^* = \begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix}}$$

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(c) (5 Points) Suppose you know that the population distribution $\boldsymbol{s}[1] = \begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix}$.

Can you determine *uniquely* the initial distribution $\boldsymbol{s}[0]$ of the population?

If your answer is "yes," explain how to compute $\boldsymbol{s}[0]$, *and* show that it's *the only* initial state that can lead to the given state $\boldsymbol{s}[1]$. If your answer is "no," provide two *distinct* initial states that lead to the same subsequent state $\boldsymbol{s}[1]$.

No! There's an uncountably-infinite set of initial states that lead to $\underline{s}[1] = \begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix}$. That's because $C = \begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix}\begin{bmatrix} 1 & 1 \end{bmatrix}$

So $C\underline{s}[0] = \begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix}\mathbb{1}^T\underline{s}[0]$. But $\mathbb{1}^T\underline{s}[0] = 1$, so $\underline{s}[1] = \begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix}$ regardless of the initial distribution of the population!

This striking result is due to the eigenvalue at zero, which makes C singular. To any valid initial state you can add a scalar multiple of $\underline{v}_2$, because $\underline{v}_2$ spans the Nullspace of C.

(d) (5 Points) Suppose you know that the population distribution $\boldsymbol{s}[3] = \begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix}$.

Can you determine *uniquely* the immediately prior distribution $\boldsymbol{s}[2]$ of the population?

If your answer is "yes," explain how to compute $\boldsymbol{s}[2]$, *and* show that it's *the only* state at $n = 2$ that can lead to the given state $\boldsymbol{s}[3]$. If your answer is "no," provide two *distinct* states at $n = 2$ that lead to the same subsequent state $\boldsymbol{s}[3]$.

Yes. Any valid initial state $\underline{s}[0]$ leads to

$$\underline{s}[1] = \underline{s}[2] = \underline{s}[3] = \cdots = \begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix}.$$

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E2.2 (35 Points) QR Decomposition and Least Squares Approximation

$\Rightarrow m \geq n$

(a) (10 Points) Suppose $\mathbf{A}$ is an $m \times n$ matrix whose columns are linearly independent and whose QR Decomposition is $\mathbf{A} = \mathbf{QR}$.

Show that the matrix $\mathbf{P_A} = \mathbf{A}(\mathbf{A}^\mathsf{T}\mathbf{A})^{-1}\mathbf{A}^\mathsf{T}$ that projects every vector $\boldsymbol{b}$ in $\mathbb{R}^n$ onto the column space of $\mathbf{A}$ is given by $\mathbf{P_A} = \mathbf{QQ}^\mathsf{T}$.

A has full col rank $\Rightarrow$ Gram-Schmidt process succeeds, and $A = QR$ is well-defined, with Q having full col rank. Also A^TA is invertible because A has full col rank.

$$A^TA = R^T \cancelto{I}{Q^TQ} R = R^TR \Rightarrow (A^TA)^{-1} = (R^TR)^{-1} = R^{-1}R^{-T}$$

$$P_A = A(A^TA)^{-1}A^T = QR(R^{-1}R^{-T})R^TQ^T = Q\cancelto{I}{RR^{-1}}\cancelto{I}{R^{-T}R^T}Q^T \Rightarrow$$

$$\boxed{P_A = QQ^T}$$

(b) (25 Points) Suppose the matrix $\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} = [\underline{a}_1 \; \underline{a}_2 \; \underline{a}_3]$

(i) (15 Points) Determine the matrices $\mathbf{Q}$ and $\mathbf{R}$ in the QR Decomposition of $\mathbf{A}$.

$$\underline{z}_1 = \underline{a}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \underline{e}_1 \Rightarrow \text{Already normalized, so } \boxed{\underline{q}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}$$

$$\underline{z}_2 = \underline{a}_2 - \langle \underline{a}_2, \underline{q}_1 \rangle \underline{q}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \underbrace{\begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{\langle \underline{a}_2, \underline{q}_1 \rangle = 1} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \underline{e}_3$$

$$\underline{z}_2 \text{ is already normalized} \Rightarrow \boxed{\underline{q}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}$$

$$\underline{z}_3 = \underline{a}_3 - \langle \underline{a}_3, \underline{q}_1 \rangle \underline{q}_1 - \langle \underline{a}_3, \underline{q}_2 \rangle \underline{q}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \underbrace{\cancelto{\langle \underline{a}_3, \underline{q}_1 \rangle = 0}{\begin{bmatrix} 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}} \underline{q}_1}_{0} - \cancelto{\langle \underline{a}_3, \underline{q}_2 \rangle = 1}{\begin{bmatrix} 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}} \underline{q}_2$$

$$= \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \underline{e}_2 \Rightarrow \underline{q}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$A = [\underline{q}_1 \; \underline{q}_2 \; \underline{q}_3] \begin{bmatrix} \|\underline{z}_1\| & \langle \underline{a}_2, \underline{q}_1 \rangle & \langle \underline{a}_3, \underline{q}_1 \rangle \\ 0 & \|\underline{z}_2\| & \langle \underline{a}_3, \underline{q}_2 \rangle \\ 0 & 0 & \|\underline{z}_3\| \end{bmatrix} \Rightarrow \boxed{A = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}}_{R}}$$

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E2.2 (b) **(Continued)**

(ii) (10 Points) Consider the over-constrained system of linear equations $\mathbf{M}\boldsymbol{x} = \boldsymbol{b}$, where

$$\mathbf{M} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \qquad \text{and} \qquad \boldsymbol{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

Determine $\widehat{\boldsymbol{b}} = \mathbf{M}\widehat{\boldsymbol{x}}$, where $\widehat{\boldsymbol{x}}$ denotes the least-squares approximate solution for the linear system of equations $\mathbf{M}\boldsymbol{x} = \boldsymbol{b}$.

Method I: Shows a solid understanding of orthogonal projections

M consists of linearly independent columns that span the subspace of vectors in $\mathbb{R}^3$ that have zero as their second entry.

This means that the orthogonal projection matrix P_M that maps any vector $\underline{b} \in \mathbb{R}^3$ onto $C(M)$, the column space of M, simply zeros out the 2nd entry of $\underline{b}$. So $\boxed{\hat{\underline{b}} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}}$

Method II: Recognize that M consists of the first two columns of A is part (a), so it has the QR decomposition

$$M = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}}_{Q_M} \underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}}_{R_M},$$ where Q_M consists of the first two columns of A, and R_M is the top-left 2x2 submatrix of the R matrix of A. Using the result of part (a), we have

$$\hat{\underline{b}} = \underbrace{Q_M Q_M^T}_{P_M} \underline{b} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$$

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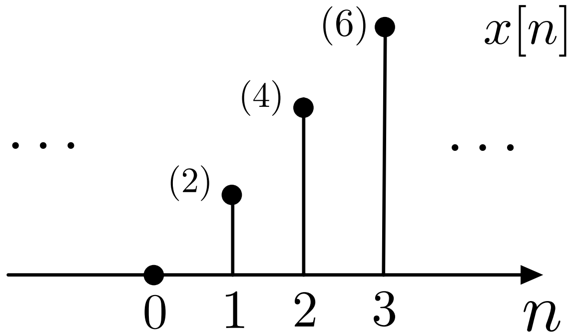
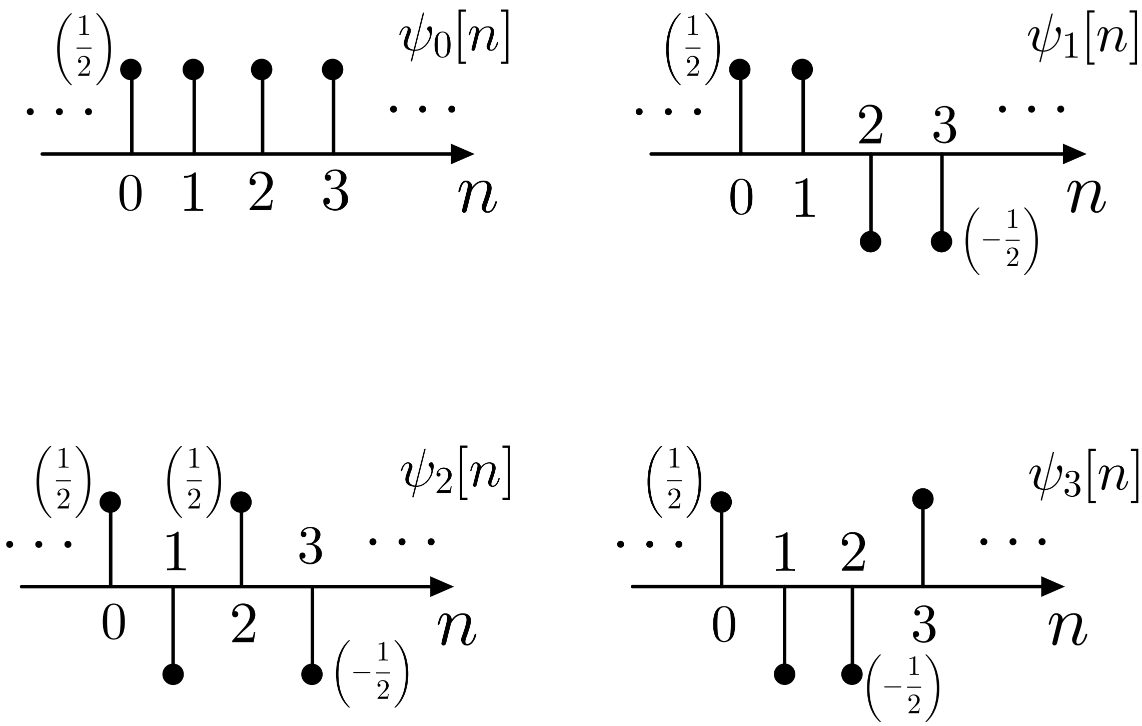
E2.3 (45 Points) Orthogonal Expansion

A single period of a 4-periodic discrete-time (DT) signal $x : \mathbb{Z} \to \mathbb{R}$ is shown below.



We want to express x as a linear combination of the signals $\{\psi_k\}_{k=0}^{3}$, each of period 4.

The figure below shows a single period of each of the orthogonal signals ψ_k.



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(a) (20 Points) Show that the signals ψ_k are *orthonormal*—that is, $\langle \psi_k, \psi_\ell \rangle = \delta[k-\ell]$, for all $k, \ell \in \{0, 1, 2, 3\}$.

$$\langle \underline{\psi}_0, \underline{\psi}_0 \rangle = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} = \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = 1$$

$$\langle \underline{\psi}_1, \underline{\psi}_1 \rangle = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{-1}{2} & \frac{-1}{2} \end{bmatrix} \begin{bmatrix} 1/2 \\ 1/2 \\ -1/2 \\ -1/2 \end{bmatrix} = \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{-1}{2}\right)^2 + \left(\frac{-1}{2}\right)^2 = 1$$

The other two, $\underline{\psi}_2$ and $\underline{\psi}_3$, produce similar sums involving $\left(\pm\frac{1}{2}\right)^2$, so the have the same unit norm.

$$\langle \underline{\psi}_0, \underline{\psi}_1 \rangle = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1/2 \\ 1/2 \\ -1/2 \\ -1/2 \end{bmatrix} = \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 = 0$$

$$\langle \underline{\psi}_0, \underline{\psi}_2 \rangle = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1/2 \\ -1/2 \\ 1/2 \\ -1/2 \end{bmatrix} = \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 = 0$$

$$\langle \underline{\psi}_0, \underline{\psi}_3 \rangle = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{bmatrix} = \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = 0$$

$$\langle \underline{\psi}_1, \underline{\psi}_2 \rangle = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{-1}{2} & \frac{-1}{2} \end{bmatrix} \begin{bmatrix} 1/2 \\ -1/2 \\ 1/2 \\ -1/2 \end{bmatrix} = \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 + \left(\frac{-1}{2}\right)^2 = 0$$

$$\langle \underline{\psi}_2, \underline{\psi}_3 \rangle = \begin{bmatrix} \frac{1}{2} & \frac{-1}{2} & \frac{1}{2} & \frac{-1}{2} \end{bmatrix} \begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{bmatrix} = \left(\frac{1}{2}\right)^2 + \left(\frac{-1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 = 0$$

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(b) (15 Points) The signal x can be decomposed into a linear combination of the orthonormal basis signals ψ_k as follows:

$$x[n] = \sum_{k=0}^{3} X_k \, \psi_k[n], \quad \text{for all } n.$$

(i) (5 Points) Without computing the coefficients X_k individually, evaluate $\sum_{k=0}^{3} X_k$.

We look for an n such that $\psi_0[n] = \psi_1[n] = \psi_2[n] = \psi_3[n] = c$

That time instance (sample) is $n=0$. So,

$$\underbrace{x[0]}_{=0} = \sum_{k=0}^{3} X_k \underbrace{\psi_k[0]}_{=1/2\ \forall k} = \frac{1}{2}\sum_{k=0}^{3} X_k \Rightarrow \boxed{\sum_{k=0}^{3} X_k = 0}$$

(ii) (10 Points) Determine the numerical values of the coefficients X_k, $k = 0, 1, 2, 3$.

Since the ψ_k's are orthonormal, this makes the calculations of the coefficients X_k simpler than otherwise:

$$x = X_0\psi_0 + X_1\psi_1 + X_2\psi_2 + X_3\psi_3$$

$$\langle x, \psi_k \rangle = \langle X_0\psi_0 + X_1\psi_1 + X_2\psi_2 + X_3\psi_3, \psi_k \rangle = X_k \cancelto{1}{\langle \psi_k, \psi_k \rangle}$$

All other terms drop out due to orthogonality. So $X_k = \langle x, \psi_k \rangle$.

$$X_0 = \langle x, \psi_0 \rangle = \begin{bmatrix} 0 & 2 & 4 & 6 \end{bmatrix} \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} = \frac{0+2+4+6}{2} = 6 \Rightarrow \boxed{X_0 = 6}$$

$$X_1 = \langle x, \psi_1 \rangle = \frac{0+2-4-6}{2} = -4 \Rightarrow \boxed{X_1 = -4}$$

$$X_2 = \langle x, \psi_2 \rangle = \frac{0-2+4-6}{2} = -2 \Rightarrow \boxed{X_2 = -2}$$

$$X_3 = \langle x, \psi_3 \rangle = \frac{0-2-4+6}{2} = 0 \Rightarrow \boxed{X_3 = 0}$$

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(c) (10 Points) Show that

$$\sum_{\cancel{k}\, n=0}^{3} |x[n]|^2 = \sum_{k=0}^{3} |X_k|^2.$$

Note: If you try to show the equality using the particular values of our signals in this problem, or those values of X_k that you computed earlier, you're likely doing more work—and hence spending more valuable time—than necessary. Try to show this result for *any* 4-periodic signal x and *any* set of 4-periodic orthonormal basis signals ψ_k. But we'll accept your solution if you plug in the particular *correct* signal values of this problem, so long as you perform the computations correctly.

$$x = \sum_{k=0}^{3} X_k \psi_k \implies \langle x, x \rangle = \left\langle \sum_{k=0}^{3} X_k \psi_k , \sum_{\ell=0}^{3} X_\ell \psi_\ell \right\rangle$$

$$= \sum_{k=0}^{3} X_k \sum_{\ell=0}^{3} X_\ell^* \underbrace{\langle \psi_k, \psi_\ell \rangle}_{\delta[k-\ell]}$$

due to orthonormality

$$= \sum_{k=0}^{3} X_k X_k^* = \sum_{k=0}^{3} |X_k|^2$$

$$\text{But } \langle x, x \rangle \triangleq \sum_{n=0}^{3} x[n] x^*[n] = \sum_{n=0}^{3} |x[n]|^2$$

$\Bigg\} \implies$

$$\boxed{\sum_{n=0}^{3} |x[n]|^2 = \sum_{k=0}^{3} |X_k|^2}$$

Parseval's Identity for our case.

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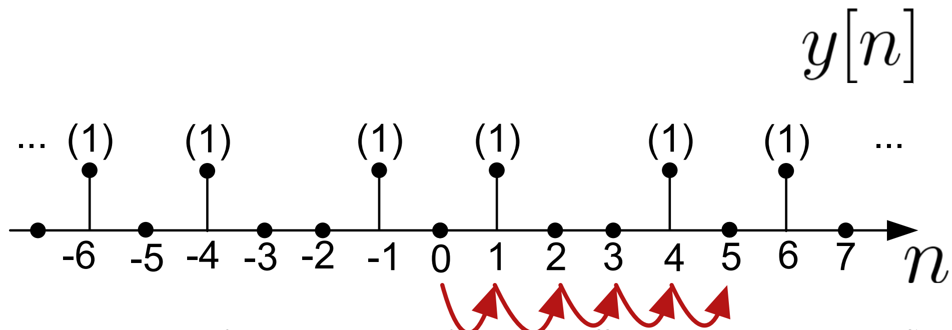
E2.4 (30 Points) DTFS I

Consider the p-periodic discrete-time signal y depicted by the figure below:



Determine a simple closed-form expression for the coefficients Y_k in the DTFS expansion

$$Y_k = \frac{1}{p} \sum_{n=\langle p \rangle} y[n]\, e^{-ik\omega_0 n}\,.$$

Needless to say, you must first determine the fundamental period p and the fundamental frequency ω_0 of y. Look carefully at the figure before you conclude what p and ω_0 are.

We note that $p = 5$, so $\omega_0 = \frac{2\pi}{5}$

$$X_k = \frac{1}{5} \sum_{n=-2}^{2} x[n] e^{-ik\omega_0 n} = \frac{1}{5}\left(\underbrace{x[-1]}_{1} e^{ik\omega_0} + \underbrace{x[1]}_{1} e^{-ik\omega_0} \right)$$

Note the judicious choice of the contiguous interval $\langle 5 \rangle = [-2, 2]$

Only two terms are nonzero, for $n = \pm 1$

$$X_k = \frac{2}{5} \frac{e^{ik\omega_0} + e^{-ik\omega_0}}{2} = \frac{2}{5}\cos(k\omega_0)$$

$\omega_0 = \frac{2\pi}{5}$ $\Rightarrow$

$$\boxed{X_k = \frac{2}{5}\cos\left(\frac{2\pi k}{5}\right)}$$

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E2.5 (40 Points) DTFS II

Consider a p-periodic discrete-time (DT) signal $x : \mathbb{Z} \to \mathbb{R}$. The signal is non-constant—that is, $p > 1$. We process the signal x by taking its N-point moving average to produce a DT signal y given by

$$\forall n \in \mathbb{Z}, \qquad y[n] = \frac{x[n] + x[n-1] + \cdots + x[n-(N-1)]}{N} = \frac{1}{N}\sum_{\ell=0}^{N-1} x[n-\ell].$$

(a) (10 Points) Show that y has period p as well—that is, $y[n+p] = y[n]$ for all n.

Note that we do not claim that p is a *fundamental period* of y, but that it is *a* period.

$$y[n+p] = \frac{1}{N}\sum_{\ell=0}^{N-1} x[n+p-\ell] = \frac{1}{N}\sum_{\ell=0}^{N-1} x[n-\ell] = y[n] \quad \forall n \in \mathbb{Z}$$

But $\forall n \in \mathbb{Z}$, $x[n+p] = x[n] \Rightarrow x[n+p-\ell] = x[n-\ell]$

$\Downarrow$

y has period p as well.

(b) (30 Points) In this part you are asked—for each of the three cases below—to determine Y_k, the DTFS coefficients of y, in terms of the corresponding coefficients X_k of the signal x:

(i) (10 Points) **Case I:** $k = 0$. Determine Y_0.

$$x[n] = \sum_{k \in \langle p \rangle} X_k e^{ik\omega_0 n} \Longrightarrow x[n-\ell] = \sum_{k \in \langle p \rangle} X_k e^{ik\omega_0 (n-\ell)} \Longrightarrow$$

$$x[n-\ell] = \sum_{k \in \langle p \rangle} X_k e^{ik\omega_0 n} e^{-ik\omega_0 \ell} \Longrightarrow y[n] = \frac{1}{N}\sum_{\ell=0}^{N-1} x[n-\ell] = \sum_{k \in \langle p \rangle} \overbrace{\left[X_k \left(\frac{1}{N}\sum_{\ell=0}^{N-1} e^{-ik\omega_0 \ell}\right)\right]}^{Y_k} e^{ik\omega_0 n}$$

$$Y_k = \frac{1 + e^{-ik\omega_0} + \cdots + e^{-ik\omega_0(N-1)}}{N} X_k$$

$$k = 0 \Longrightarrow Y_0 = \frac{\overbrace{1 + \cdots + 1}^{N \text{ terms}}}{N} X_0 \Longrightarrow \boxed{Y_0 = X_0}$$

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E2.5 (b) **(Continued)**

(ii) (10 Points) **Case II:** $k \neq 0$ and $N = p$. Determine Y_k.

If $k \neq 0$ & $N = p$:

$$\boxed{Y_k = 0}$$

$$N = p \Rightarrow Y_k = \frac{1 + e^{-ik\omega_0} + \cdots + e^{-ik\omega_0(p-1)}}{p} X_k$$

$$= \frac{1 + e^{-ik\frac{2\pi}{p}} + \cdots + e^{-ik\frac{2\pi}{p}(p-1)}}{p} X_k$$

The numerator is zero. We can see this by either noting that it's the sum of the p p^{th} roots of 1, or using the geometric sum formula $\sum_{l=L}^{U} \alpha^l = \begin{cases} U-L+1 & \text{if } \alpha = 1 \\ \frac{\alpha^{U+1} - \alpha^L}{\alpha - 1} & \text{if } \alpha \neq 1 \end{cases}$

(iii) (10 Points) **Case III:** $k \neq 0$ and $1 \leq N < p$. Determine Y_k.

If $k \neq 0$ and $1 \leq N < p$, then we can apply the geometric sum formula

$$\sum_{l=L}^{U} \alpha^l = \frac{\alpha^{U+1} - \alpha^L}{\alpha - 1} \quad \text{with} \begin{cases} L = 0 \\ U = N-1 \\ \alpha = e^{-ik\omega_0} \end{cases}$$

$$Y_k = \frac{1 + e^{-ik\frac{2\pi}{p}} + \cdots + e^{-ik\frac{2\pi}{p}(N-1)}}{N} X_k$$

$$= \frac{1}{N} \frac{e^{-ik\omega_0 N} - 1}{e^{-ik\omega_0} - 1} X_k \Rightarrow \boxed{Y_k = \frac{1}{N} \frac{1 - e^{-ik\omega_0 N}}{1 - e^{-ik\omega_0}} X_k}$$