EECS 16A: Foundations of Signals, Dynamical Systems, and Information Processing Exam 3

Department of Electrical Engineering and Computer Sciences

UNIVERSITY OF CALIFORNIA, BERKELEY 19 December 2025

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- **(5 Points)** On *every* page, print legibly your name and ALL digits of your SID. For every page on which you do not write your name and SID, you forfeit a point, up to the maximum 5 points.
- **(10 Points) (Pledge of Academic Integrity)** Hand-copy, sign, and date the single-line text (which begins with *I have read, ...*) of the Pledge of Academic Integrity on page 3 of this document. Your solutions will *not* be evaluated without this.
- **Urgent Contact with the Teaching Staff:** In case of an urgent matter, raise your hand.
- **This document consists of pages numbered 1 through 16.** Verify that your copy of the exam is free of anomalies, and contains all of the specified number of pages. If you find a defect in your copy, contact the teaching staff immediately.
- This exam is designed to be completed within 110 minutes. However, you may use up to 120 minutes total—*in one sitting*—to tackle the exam.

  The exam starts at 3:10 pm California time. Your allotted window begins with respect to this start time. Students who have official accommodations of 1.5× and 2× time windows have 180 and 240 minutes, respectively.
- **This exam is closed book.** You may not use or access, or cause to be used or accessed, any reference in print or electronic form at any time during the exam, except three double-sided 8.5"×11" sheets of handwritten, original notes having no appendage.

  Collaboration is <u>not</u> permitted.

  Computing, communication, and other electronic devices (except dedicated timekeepers) must be turned off.

  Scratch paper will be provided to you; ask for more if you run out. You may not use your own scratch paper.
- Please write neatly and legibly, because *if we can't read it, we can't evaluate it.*
- For each problem, limit your work to the space provided specifically for that problem. *No other work will be considered. For example, we will not evaluate scratch work. No exceptions.*
- Unless explicitly waived by the specific wording of a problem, you must explain your responses (and reasoning) succinctly, but clearly and convincingly.
- In some parts of a problem, we may ask you to establish a certain result—for example, "show this" or "prove that." Even if you're unable to establish the result that we ask of you, you may still take that result for granted—and use it in any subsequent part of the problem.

- If we ask you to provide a "reasonably simple expression" for something, by default we expect your expression to be in closed form—one *not* involving a sum $\sum$ or an integral $\int$—*unless* we explicitly tell you otherwise.
- Noncompliance with these or other instructions from the teaching staff—*including, for example, commencing work prematurely, or continuing it beyond the allocated time window*—is a serious violation of the Code of Student Conduct.

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Pledge of Academic Integrity

By my honor, I affirm that

(1) this document—which I have produced for the evaluation of my performance—reflects my original, bona fide work, and that I have neither provided to, nor received from, anyone excessive or unreasonable assistance that produces unfair advantage for me or for any of my peers;

(2) as a member of the UC Berkeley community, I have acted with honesty, integrity, respect for others, and professional responsibility—and in a manner consistent with the letter and intent of the campus Code of Student Conduct;

(3) I have not violated—nor aided or abetted anyone else to violate—the instructions for this exam given by the course staff, including, but not limited to, those on the cover page of this document; and

(4) More generally, I have not committed any act that violates—nor aided or abetted anyone else to violate—UC Berkeley, state, or Federal regulations, during this exam.

(10 Points) In the space below, hand-write the following sentence, verbatim. Then write your name in legible letters, sign, include your full SID, and date before submitting your work:

I have read, I understand, and I commit to adhere to the letter and spirit of the pledge above.

I have read, I understand, and I commit to adhere to the letter and spirit of the pledge above.

Full Name: Singalar Viktar

Signature:

Date: 19 Dec 2025

Student ID: 1234567890

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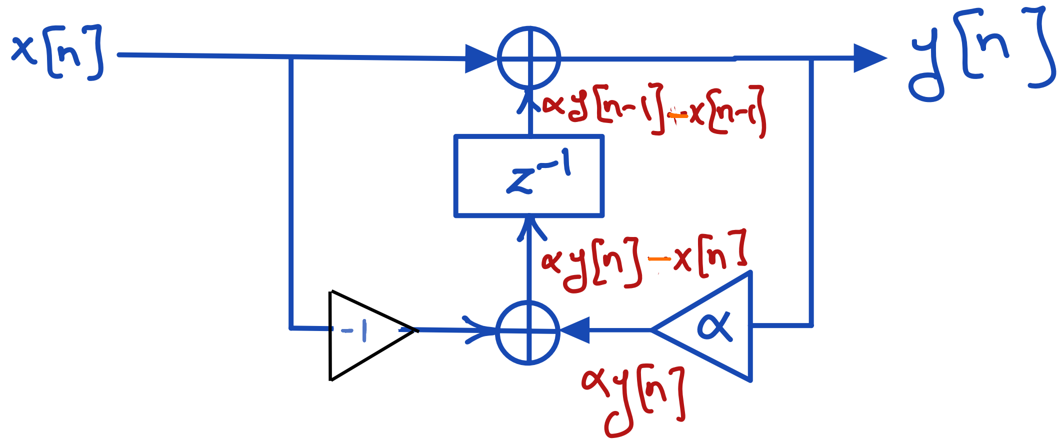
E3.1 (70 Points) DT-LTI Filtering

The input-output behavior of a single-input single-output discrete-time LTI filter F is described by the following linear, constant-coefficient difference equation:

$$\forall n \in \mathbb{Z}, \quad y[n] = \alpha\, y[n-1] + x[n] - x[n-1],$$

where $0 < \alpha < 1$, and $y[n] = 0$ for all $n < 0$.

(a) (10 Points) Provide a *well-labeled* delay-adder-gain block diagram implementation of the filter. Your implementation must use the minimal number of delay blocks needed. You need *not* provide *any* explanation for your diagram.



(b) (5 Points) Explain why this filter *must* be IIR, even in the absence of knowing the expression for its impulse response. A sufficient answer could be a simple "one-liner."

The LCCDE is recursive: $y[n]$ depends at least in part on $y[n-1]$.

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(c) (10 Points) In this part we explore the impulse response of the filter.

(i) (5 Points) Show that the impulse response $f[n]$ is expressible in one of the following equivalent forms:

$$\forall n \in \mathbb{Z}, \qquad f[n] = \delta[n] - (1-\alpha)\alpha^{n-1}u[n-1] = \alpha^n u[n] - \alpha^{n-1}u[n-1],$$

where u denotes the discrete-time unit step function.

$$f[n] = \alpha h[n-1] + \delta[n] - \delta[n-1]$$

$$f[0] = \alpha \underbrace{h[-1]}_{0} + \delta[0] - \underbrace{\delta[-1]}_{0} = 1$$

$$f[1] = \alpha \underbrace{h[0]}_{1} + \underbrace{\delta[1]}_{0} - \underbrace{\delta[0]}_{1} = \alpha - 1$$

$$f[2] = \alpha h[1] = \alpha^2 - \alpha = -(1-\alpha)\alpha$$

$$f[3] = \alpha h[2] = \alpha^3 - \alpha^2 = -(1-\alpha)\alpha^2$$

$$\vdots$$

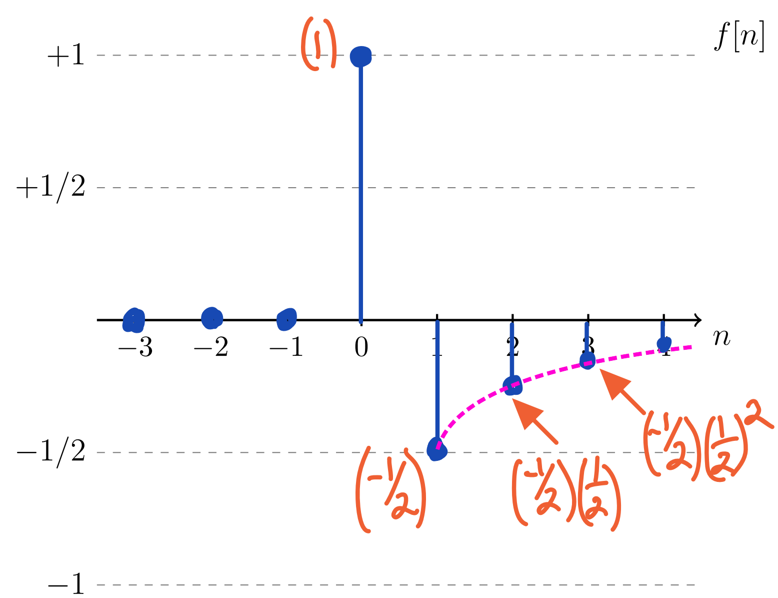
$$f[n] = \alpha^n - \alpha^{n-1} = -(1-\alpha)\alpha^{n-1}$$

$$f[n] = \begin{cases} 0 & n < 0 \\ 1 & n = 0 \\ -(1-\alpha)\alpha^{n-1} & n \geq 1 \end{cases}$$

$$= \delta[n] - (1-\alpha)\alpha^{n-1}u[n-1]$$

$$= \alpha^n u[n] - \alpha^{n-1}u[n-1]$$

(ii) (5 Points) *For this subpart only*, assume $\alpha = 1/2$ and provide a well-labeled plot of the impulse response f. Be sure to mark those values of n for which $f[n] = 0$.



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(d) (10 Points) Show that the system F is BIBO stable for $0 < \alpha < 1$. Use *either* (but *not* both) of the following two methods:

(I) Determine a finite numerical value for $\|f\|_1$; or

(II) Determine a finite upper bound for $\|f\|_1$,

where $\|f\|_1 \triangleq \sum_{n=-\infty}^{\infty} |f[n]|$ denotes the 1-norm of f.

It may or may not be convenient to recall that $\sum_{n=0}^{\infty} \lambda^n = \frac{1}{1-\lambda}$ if $|\lambda| < 1$.

Method I

$$f[n] = \delta[n] - (1-\alpha)\alpha^{n-1} u[n-1]$$

$$\|f\|_1 = \sum_{n=-\infty}^{\infty} |f[n]| = \underbrace{|f[0]|}_{1} + \sum_{n=1}^{\infty} |-(1-\alpha)\alpha^{n-1}| = 1 + \sum_{n=1}^{\infty} (1-\alpha)\alpha^{n-1}$$

$$\boxed{0 < \alpha < 1 \Rightarrow 0 < 1-\alpha}$$

$$\|f\|_1 = \sum_{n=-\infty}^{\infty} |f[n]| = 1 + (1-\alpha)\sum_{n=1}^{\infty} \alpha^{n-1} = 1 + (1-\alpha)\sum_{l=0}^{\infty} \alpha^{l} = 1 + \cancel{(1-\alpha)}\frac{1}{\cancel{1-\alpha}}$$

Let $l = n-1$

$$\Longrightarrow \|f\|_1 = \sum_{n=-\infty}^{\infty} |f[n]| = 2 < \infty$$

Impulse response is absolutely summable, so the system is BIBO stable.

Method II:

$$f[n] = \alpha^n u[n] - \alpha^{n-1} u[n-1] \Longrightarrow |f[n]| \leq |\alpha^n u[n]| + |\alpha^{n-1} u[n-1]| \Longrightarrow$$

$$\|f\|_1 = \sum_{n=-\infty}^{\infty} |f[n]| \leq \sum_{n=-\infty}^{\infty} |\alpha^n u[n]| + \sum_{n=-\infty}^{\infty} |\alpha^{n-1} u[n-1]| = \sum_{n=0}^{\infty} \alpha^n + \sum_{n=1}^{\infty} \alpha^{n-1} < \infty$$

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(e) (20 Points) In this part we explore aspects related to the frequency response of the filter F.

(i) (10 Points) Show that the frequency response of the filter is given by

$$F(\omega) = \frac{e^{i\omega} - 1}{e^{i\omega} - \alpha}.$$

Method I: Eigenfunction Property

$$y[n] = \alpha y[n-1] + x[n] - x[n-1]$$

By the eigenfunction property of complex exponentials with respect to DT-LTI systems.

Let $x[n] = e^{i\omega n} \Rightarrow y[n] = F(\omega)e^{i\omega n}$

$\Rightarrow x[n-1] = e^{i\omega(n-1)} = e^{-i\omega}e^{i\omega n}$

& $y[n-1] = F(\omega)e^{i\omega(n-1)} = e^{-i\omega}F(\omega)e^{i\omega n}$

Plug these into the LCCDE, and solve for $F(\omega)$:

$$\underbrace{F(\omega)\cancel{e^{i\omega n}}}_{y[n]} = \underbrace{\alpha e^{-i\omega}F(\omega)\cancel{e^{i\omega n}}}_{\alpha y[n-1]} + \underbrace{\cancel{e^{i\omega n}}}_{x[n]} - \underbrace{e^{-i\omega}\cancel{e^{i\omega n}}}_{x[n-1]}$$

$$F(\omega) = \alpha e^{-i\omega}F(\omega) + 1 - e^{-i\omega} \Rightarrow (1-\alpha e^{-i\omega})F(\omega) = 1 - e^{-i\omega}$$

$$\Rightarrow F(\omega) = \frac{1-e^{-i\omega}}{1-\alpha e^{-i\omega}} = \frac{e^{i\omega}-1}{e^{i\omega}-\alpha}$$

Method II: Definition of Frequency Response: $F(\omega) = \sum_{n=-\infty}^{\infty} f[n]e^{-i\omega n}$

$$f[n] = \alpha^n u[n] - \alpha^{n-1}u[n-1] \Rightarrow F(\omega) = \sum_{n=0}^{\infty}\alpha^n e^{-i\omega n} - \sum_{n=1}^{\infty}\alpha^{n-1}e^{-i\omega n}$$

$$F(\omega) = \sum_{n=0}^{\infty}(\alpha e^{-i\omega})^n - e^{-i\omega}\sum_{n=1}^{\infty}\alpha^{n-1}e^{-i\omega(n-1)} = \sum_{n=0}^{\infty}(\alpha e^{-i\omega})^n - e^{-i\omega}\sum_{\ell=0}^{\infty}(\alpha e^{-i\omega})^\ell$$

$$0<\alpha<1 \Rightarrow 0<|\alpha e^{-i\omega}|<1 \Rightarrow F(\omega) = \frac{1}{1-\alpha e^{-i\omega}} - \frac{e^{-i\omega}}{1-\alpha e^{-i\omega}} = \frac{1-e^{-i\omega}}{1-\alpha e^{-i\omega}}$$

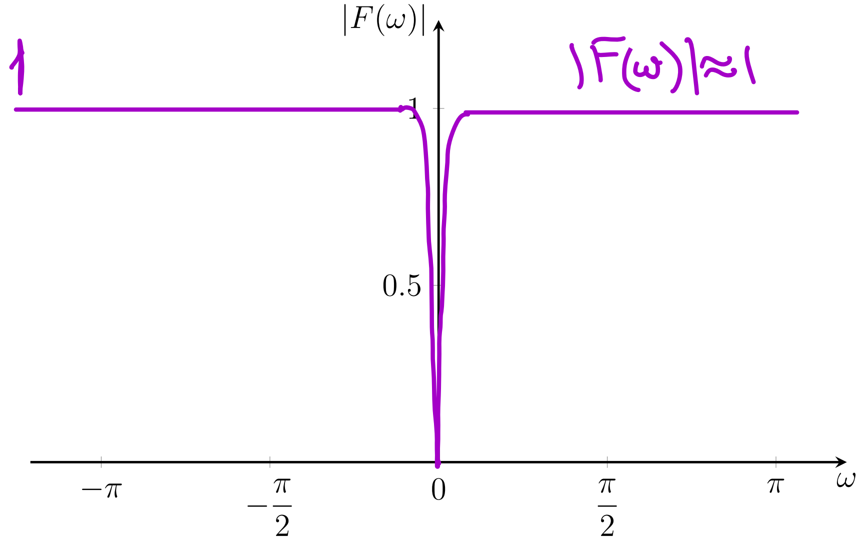
$$\Rightarrow F(\omega) = \frac{e^{i\omega}-1}{e^{i\omega}-\alpha}$$

E3.1 (Continued)

(ii) (10 Points) Assume for this part that $\alpha = 0.99$. Provide a well-labeled *approximate* plot of $|F(\omega)|$, the magnitude response of the filter. Use the frequency interval $-\pi \leq \omega \leq \pi$ for your plot.

Do *not* use complicated expressions. Instead, use a geometric approach like that shown in lecture.

Also, use reasonable engineering approximations. For example, if p and q are two points on the complex plane that are very close to each other, and r is a third point *not* in a small neighborhood of p and q, then you may safely assume that $|r - p| \approx |r - q|$.



$$|F(\omega)| = \frac{|e^{i\omega} - 1|}{|e^{i\omega} - 0.99|}$$

For all frequencies not in the small neighborhood of $\omega = 0$, the numerator and denominator are approximately equal: $|F(\omega)| \approx 1$ for all ω not near zero. For $\omega = 0$, $F(\omega) = 0$, so $|F(0)| = 0$. As $\omega \to 0^+$ or 0^-, the numerator tends to zero, but the denominator tends to 0.01, so $|F(\omega)|$ dips down from either side of $\omega = 0$. This is an example of a Notch Filter, as shown above.

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(f) (10 Points) Let the input to the filter be

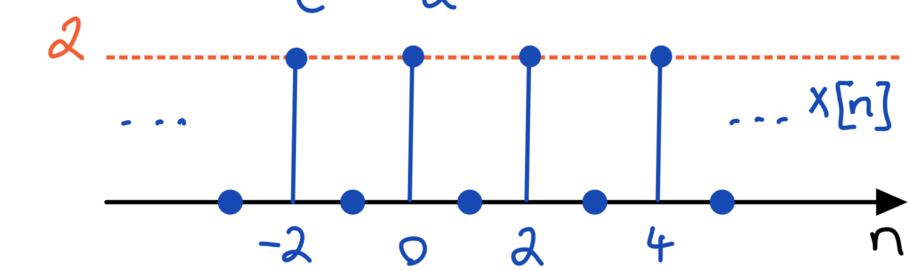
$$\forall n \in \mathbb{Z}, \quad x[n] = 1 + (-1)^n.$$

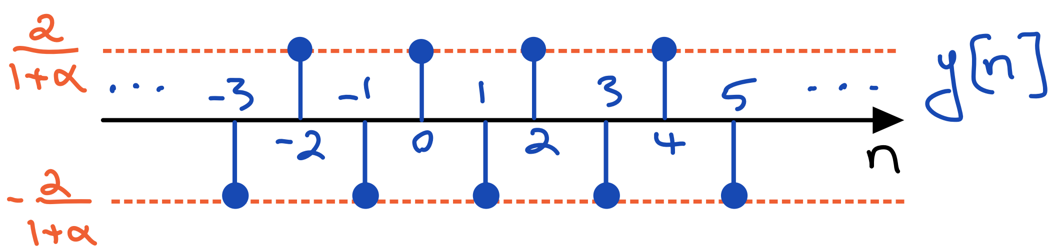
Determine a reasonably simple closed-form expression for the output $y[n]$ for all $n \in \mathbb{Z}$.

Hint: Think in terms of the eigenfunction property of complex exponentials with respect to DT-LTI systems.

$$x[n] = 1 + (-1)^n = e^{i0n} + e^{i\pi n} \implies y[n] = \cancel{F(0)}_{0}\, e^{i0n} + F(\pi)e^{i\pi n}$$

$$F(\pi) = \frac{e^{i\pi} - 1}{e^{i\pi} - \alpha} = \frac{-1-1}{-1-\alpha} = \frac{2}{1+\alpha} \implies \boxed{y[n] = \frac{2}{1+\alpha}(-1)^n}$$





(g) (5 Points) For this part, assume $y[0] = Y_0$ for some nonzero Y_0. Determine a reasonably simple expression for $y_{\mathsf{ZIR}}[n]$, the zero-input response of the filter. Naturally, you must assume $x[n] = 0$ for all integers n.

$$y[n] = \alpha y[n-1] \qquad y[0] = Y_0$$

$$y[1] = \alpha y[0] = \alpha Y_0$$

$$y[2] = \alpha y[1] = \alpha^2 Y_0$$

$$\vdots$$

$$\boxed{y[n] = \alpha^n Y_0} \qquad n \geq 0$$

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E3.2 (45 Points) From LCCDE to State Space

The input-output behavior of a single-input single-output discrete-time LTI filter G is described by the following *third-order* linear, constant-coefficient difference equation:

$$\forall n \in \mathbb{Z}, \quad y[n] + a_1\, y[n-1] + a_2\, y[n-2] + a_3\, y[n-3] = x[n],$$

where a_1, a_2, and a_3 are nonzero real coefficients.

(a) (20 Points) Choose as your first state variable $q_1[n] = y[n-3]$. With a proper choice of the other two state variables $q_2[n]$ and $q_3[n]$, show that an appropriate state-evolution equation that can represent the system's dynamic behavior is given by

$$\boldsymbol{q}[n+1] = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix}}_{\mathbf{A}} \boldsymbol{q}[n] + \mathbf{B}x[n],$$

where $\boldsymbol{q}[n] = \begin{bmatrix} q_1[n] & q_2[n] & q_3[n] \end{bmatrix}^{\mathsf{T}}$ denotes the state vector at sample n. Be sure to determine the input matrix $\mathbf{B}$ as part of your work.

$$y[n] + a_1 \underbrace{y[n-1]}_{q_3[n]} + a_2 \underbrace{y[n-2]}_{q_2[n]} + a_3 \underbrace{y[n-3]}_{q_1[n]} = x[n]$$

$$q_1[n+1] = y[n-2] = q_2[n] = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \underline{q}[n] + 0 \cdot x[n]$$

$$q_2[n+1] = y[n-1] = q_3[n] = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \underline{q}[n] + 0 \cdot x[n]$$

$$q_3[n+1] = y[n] = -a_1 y[n-1] - a_2 y[n-2] - a_3 y[n-3] + x[n]$$

$$= -a_1 q_3[n] - a_2 q_2[n] - a_3 q_1[n] + x[n]$$

$$= \begin{bmatrix} -a_3 & -a_2 & -a_1 \end{bmatrix} \underline{q}[n] + 1\, x[n]$$

$$\begin{bmatrix} q_1[n+1] \\ q_2[n+1] \\ q_3[n+1] \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix} \begin{bmatrix} q_1[n] \\ q_2[n] \\ q_3[n] \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} x[n]$$

(b) (10 Points) Provide the matrices $\mathbf{C}$ and $\mathbf{D}$ in the output equation of the system:

$$y[n] = \mathbf{C}\boldsymbol{q}[n] + \mathbf{D}x[n].$$

$$y[n] = q_3[n+1] = \underbrace{\begin{bmatrix} -a_3 & -a_2 & -a_1 \end{bmatrix}}_{C} \underline{q}[n] + \underbrace{1}_{D} \cdot x[n]$$

(c) (15 Points) Suppose the input is zero (i.e., $x[n] = 0$ for all $n \in \mathbb{Z}$); the initial state is $\boldsymbol{q}[0] = \mathbf{1} \in \mathbb{R}^3$; and $1 + a_1 + a_2 + a_3 = 0$.

Determine a simple closed-form expression for the state vector $\boldsymbol{q}[n]$ for all $n \geq 0$.

The constraint $1 + a_1 + a_2 + a_3 = 0$ means $-a_1 - a_2 - a_3 = 1$, which makes the state-transition matrix A row stochastic. This means $\underline{v} = \mathbb{1}$ is an eigenvector of A, with corresponding eigenvalue $\lambda = 1$.

$A\mathbb{1} = \mathbb{1}$.

Since the input is zero, $\underline{q}[n] = A^n \underline{q}[0]$ for all $n \geq 0$.

Clearly, $\underline{q}[0] = \underline{v} = \mathbb{1} \implies \underline{q}[n] = \mathbb{1} \quad \forall n \geq 0$.

In other words, the system starts in an equilibrium state, so it stays in that equilibrium state forever (given $x[n] = 0 \; \forall n$)

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E3.3 (70 Points) SVD Image Compression

Consider an $m \times n$ grayscale image represented by a data matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$. Its (economy) singular value decomposition (SVD) is

$$\mathbf{A} = \mathbf{U}_r \mathbf{\Sigma}_r \mathbf{V}_r^\mathsf{T} = \sum_{\ell=1}^{r} \sigma_\ell \, \boldsymbol{u}_\ell \boldsymbol{v}_\ell^\mathsf{T}, \tag{1}$$

where

- $r = \operatorname{rank}(\mathbf{A})$;
- $\mathbf{U}_r = \begin{bmatrix} \boldsymbol{u}_1 & \cdots & \boldsymbol{u}_\ell & \cdots & \boldsymbol{u}_r \end{bmatrix} \in \mathbb{R}^{m\times r}$ is the economy matrix of left singular vectors;
- $\mathbf{V}_r = \begin{bmatrix} \boldsymbol{v}_1 & \cdots & \boldsymbol{v}_\ell & \cdots & \boldsymbol{v}_r \end{bmatrix} \in \mathbb{R}^{n\times r}$ is the economy matrix of right singular vectors;
- $\boldsymbol{u}_\ell \in \mathbb{R}^m$ are orthonormal left singular vectors;
- $\boldsymbol{v}_\ell \in \mathbb{R}^n$ are orthonormal right singular vectors; and
- $\sigma_1 \ge \sigma_2 \ge \cdots \ge \sigma_\ell \ge \cdots \ge \sigma_r > 0$ are the nonzero singular values.

Consider the rank-one matrices

$$\mathbf{W}_\ell \triangleq \boldsymbol{u}_\ell \boldsymbol{v}_\ell^\mathsf{T}, \qquad \ell = 1, \ldots, r,$$

as well as the following inner product defined on the vector space of $m \times n$ real matrices:

$$\langle \mathbf{P}, \mathbf{Q} \rangle \triangleq \operatorname{tr}(\mathbf{P}\mathbf{Q}^\mathsf{T}), \qquad \mathbf{P}, \mathbf{Q} \in \mathbb{R}^{m\times n}.$$

(a) (10 Points) Explain why the inner product defined above satisfies commutativity—that is, $\langle \mathbf{P}, \mathbf{Q} \rangle = \langle \mathbf{Q}, \mathbf{P} \rangle$.

$\langle P, Q \rangle = \operatorname{tr}(PQ^T)$ But $QP^T = (PQ^T)^T$, and

$\langle Q, P \rangle = \operatorname{tr}(QP^T)$

we know that trace is invariant under transposition,

so $\underbrace{\operatorname{tr}(QP^T)}_{\langle Q, P \rangle} = \operatorname{tr}((PQ^T)^T) = \underbrace{\operatorname{tr}(PQ^T)}_{\langle P, Q \rangle}$

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(b) (15 Points) Show that the rank-one matrices $\mathbf{W}_\ell = \boldsymbol{u}_\ell \boldsymbol{v}_\ell^\mathsf{T}$ are orthonormal with respect to the matrix inner product defined above—that is,

$$\langle \mathbf{W}_\ell, \mathbf{W}_j \rangle = \delta[\ell - j] \qquad (1 \le \ell, j \le r),$$

where δ is the Kronecker delta. Note that you must establish two facts here—namely,

(I) Orthogonality: $\mathbf{W}_\ell \perp \mathbf{W}_j$ (i.e., $\langle \mathbf{W}_\ell, \mathbf{W}_j \rangle = 0$) if $\ell \neq j$; and

(II) Normalization: $\|\mathbf{W}_\ell\|^2 \triangleq \langle \mathbf{W}_\ell, \mathbf{W}_\ell \rangle = 1$ for all $1 \le \ell \le r$.

$$\langle W_\ell, W_j \rangle = tr(W_\ell W_j^T) = tr\left(\underline{u}_\ell \underline{v}_\ell^T (\underline{u}_j \underline{v}_j^T)^T\right) = tr\left(\underline{u}_\ell \underbrace{\underline{v}_\ell^T \underline{v}_j}_{\delta[\ell-j]} \underline{u}_j^T\right)$$

$$= \begin{cases} 0 & \text{if } \ell \neq j \\ tr(\underline{u}_\ell \underline{u}_\ell^T) & \text{if } \ell = j \end{cases}$$

But $tr(\underline{u}_\ell \underline{u}_\ell^T) = \|\underline{u}_\ell\|^2 = 1$, so $\langle W_\ell, W_j \rangle = \delta[\ell - j]$ $1 \le \ell, j \le r$

Normalization of $\|W_\ell\|^2$ is a consequence of the normalization of the $\underline{u}_\ell$'s.

(c) (15 Points) Based on the previous results, we can think of the SVD of $\mathbf{A}$ as an *orthogonal expansion* of the form

$$\mathbf{A} = \sigma_1 \mathbf{W}_1 + \cdots + \sigma_\ell \mathbf{W}_\ell + \cdots + \sigma_r \mathbf{W}_r, \tag{2}$$

where each element $\mathbf{W}_\ell$ in the orthonormal basis $\{\mathbf{W}_\ell\}_{\ell=1}^r$ spans one of the mutually orthogonal "directions" in the space of $m \times n$ real matrices.

Express the ℓ^{th} coefficient σ_ℓ in terms of an inner product, and provide a geometric interpretation for it. Think in terms of projections.

Standard ~~Trick~~ (technique) for orthogonal expansions: Project the left-hand side onto the ℓ^{th} basis element (in this case W_ℓ):

$$\langle A, W_\ell \rangle = \langle \sigma_1 W_1 + \cdots + \sigma_r W_r, W_\ell \rangle = \sigma_\ell \cancelto{1}{\langle W_\ell, W_\ell \rangle} \Longrightarrow$$

$$\boxed{\sigma_\ell = \langle A, W_\ell \rangle}$$

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(d) (10 Points) Suppose we want to approximate the image by keeping only $k \ll r$ terms in the orthogonal expansion (2). Which k terms provide the best rank-k approximation $\widehat{\mathbf{A}}_k$ to $\mathbf{A}$ in the sense of the Frobenius norm? That is, we want

$$\widehat{\mathbf{A}}_k \triangleq \underset{\text{rank}(\widehat{\mathbf{A}})=k}{\text{argmin}} \|\mathbf{A} - \widehat{\mathbf{A}}\|_F.$$

Express $\widehat{\mathbf{A}}_k$ in terms of a linear combination of a subset of the orthonormal basis $\{\mathbf{W}_\ell\}_{\ell=1}^r$.

Though you don't need to use it here explicitly, the Frobenius norm of an $m \times n$ real matrix $\mathbf{P}$ is $\|\mathbf{P}\|_F \triangleq \sqrt{\sum_{\substack{1\le i\le m \\ 1\le j\le n}} p_{ij}^2} = \sqrt{\operatorname{tr}(\mathbf{P}^\top \mathbf{P})}$. In other words, $\|\mathbf{P}\|_F^2$ is the sum of the squares of the entries of $\mathbf{P}$.

By the Eckart-Young-Mirsky Theorem

$$\hat{A}_k = \sum_{\ell=1}^{k} \sigma_\ell W_\ell$$

(e) (5 Points) You are given the following singular values of a matrix $\mathbf{A}$:

$\sigma_1 = 300, \sigma_2 = 120, \sigma_3 = 35, \sigma_4 = 9,\ \sigma_5 = 2, \sigma_6 = 0.6, \sigma_7 = 0.3, \sigma_8 = 0.1, \sigma_9 = 0.05,\ \ldots.$

Which value of k would you choose for $\widehat{\mathbf{A}}_k$ to retain most of the meaningful structure of the image?

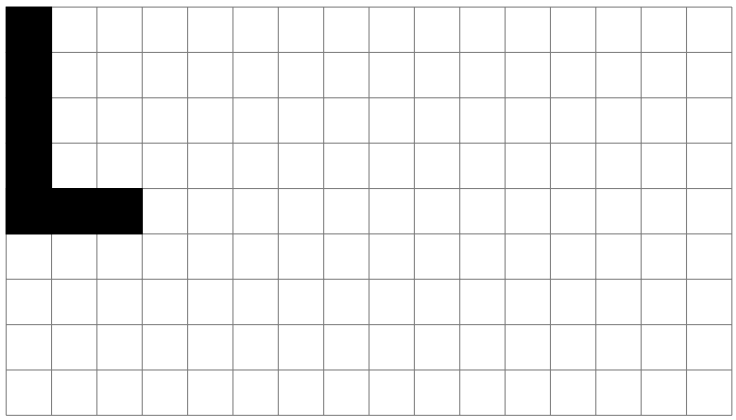
$$\hat{A}_3 = \sum_{\ell=1}^{3} \sigma_\ell W_\ell \quad \text{or} \quad \hat{A}_2 = \sum_{\ell=1}^{2} \sigma_\ell W_\ell$$

The most significant singular values are σ_1 & σ_2, so keeping those terms retains most of the structure.

(f) (15 Points) Now consider the following 9×16 grayscale image matrix $\mathbf{A}$, defined by

$$a_{ij} = \begin{cases} 1, & \text{if } (j = 1 \text{ and } 1 \leq i \leq 5) \text{ or } (i = 5 \text{ and } 1 \leq j \leq 3), \\ 0, & \text{otherwise,} \end{cases}$$

where $1 \leq i \leq 9$ and $1 \leq j \leq 16$. Here 1 represents a dark pixel and 0 represents a light pixel. Thus, $\mathbf{A}$ depicts a bold dark "L" shape: a vertical bar in column 1 (rows 1–5) and a horizontal bar in row 5 (columns 1–3).



You are given the following rank-1 approximation to $\mathbf{A}$:

$$\widehat{\mathbf{A}}_1 = \sigma_1 \mathbf{W}_1,$$

where $\widehat{\mathbf{A}}_1$ is explicitly given by

$$\widehat{\mathbf{A}}_1 = \begin{bmatrix} 0.86 & 0.24 & 0.24 & 0 & \cdots & 0 \\ 0.86 & 0.24 & 0.24 & 0 & \cdots & 0 \\ 0.86 & 0.24 & 0.24 & 0 & \cdots & 0 \\ 0.86 & 0.24 & 0.24 & 0 & \cdots & 0 \\ 1.35 & 0.38 & 0.38 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}_{9 \times 16}.$$

Due to decimal truncation to keep the printed size reasonable, the matrix shown above does not appear to have rank 1. Ignore this artifact. The actual matrix $\mathbf{A}_1$ does have rank 1.

For the remaining questions, do *not* attempt to compute the SVD. Instead, reason qualitatively.

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(i) (5 Points) Inspect the matrix $\widehat{\mathbf{A}}_1$ above. Which major geometric feature of the original "L"-shaped image is captured by this rank-1 approximation? Your explanation should refer to what kinds of image structure a single outer product $\boldsymbol{u}\boldsymbol{v}^\mathsf{T}$ can represent.

The vertical portion of the L in rows 1 to 5. This is the dominant structural component (feature) of L. We can see it "smeared" horizontally, something that subsequent terms in the rank-1 expansion of A would correct— without altering the first term (due to orthoganality).

(ii) (10 Points) How many rank-one terms from the orthogonal expansion $\mathbf{A} = \sum_{\ell=1}^{r} \sigma_\ell \boldsymbol{u}_\ell \boldsymbol{v}_\ell^\mathsf{T}$ are required to reconstruct $\mathbf{A}$ *exactly*? In other words, determine k such that the rank-k approximation

$$\widehat{\mathbf{A}}_k \triangleq \sigma_1 \mathbf{W}_1 + \cdots + \sigma_k \mathbf{W}_k$$

is equal to $\mathbf{A}$.

Exactly two terms. The original image A is a rank-2 matrix (the first two columns of A span the column space of A). So

$A = \sigma_1 W_1 + \sigma_2 W_2$ exactly.