EECS 16A Designing Information Devices and Systems I
Spring 2020

Midterm 1

Midterm 1 Solution

PRINT your student ID: ________________________________

PRINT AND SIGN your name: ________________, ________________ ________________
(last name) (first name) (signature)

PRINT your discussion section and GSI(s) (the one you attend): ________________________________

Name and SID of the person to your left: ________________________________

Name and SID of the person to your right: ________________________________

Name and SID of the person in front of you: ________________________________

Name and SID of the person behind you: ________________________________

1. What are you looking forward to over Spring Break? (3 points)

2. Approximately what % of lectures do you watch regularly, either online or in person? (0 points)
For statistical purposes only.

◯ 0% ◯ 25% ◯ 50% ◯ 75% ◯ 100%

3. Tell us about something that makes you happy. (3 points)

Do not turn this page until the proctor tells you to do so. You may work on the questions above.

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4. Splotchy Writing (10 points)

Professor Courtade writes with a sharpie to accommodate the vision of as many people as possible. Unfortunately, some characters get smudged, which makes them difficult to read. The following is a (hypothetical) passage from lecture notes, and the smudges are labeled $\textcircled{1}, \textcircled{2}, \ldots, \textcircled{10}$. Your task is to identify correct expressions for each of the smudges.

Let $A \in \mathbb{R}^{n \times \textcircled{1}}$ be a matrix with rank r. It is always possible to write A in terms of its compact SVD

$$A = U\Sigma V^\top,$$

where Σ is a diagonal $r \times r$ matrix, and $U \in \mathbb{R}^{\textcircled{2} \times \textcircled{3}}$ and $V \in \mathbb{R}^{\textcircled{4} \times \textcircled{5}}$ have orthonormal columns. This means that $U^\top U = I_{\textcircled{6}}$ and $V^\top V = I_{\textcircled{7}}$, where we write I_m to denote the $m \times \textcircled{8}$ identity matrix, for an integer m. The columns of U form a basis for the range of A, which is is defined as

$$\text{range}(A) = \{A\vec{x} \mid \vec{x} \in \mathbb{R}^k\}.$$

Note that range(A) is a subspace of $\mathbb{R}^{\textcircled{9}}$, which has dimension $\textcircled{10}$.

Select the values for each smudge from the multiple choice below. For each smudge, completely fill in the circle next to the correct answer. (Hint: Resist the temptation to get distracted by unfamiliar terminology... that isn't what this question is about.)

Concepts: This question tests your understanding of matrix multiplication (specifically, compatibility of dimensions necessary for, and resulting from, matrix-matrix multiplication), as well as dimension of column-space (i.e., range). As suggested by the hint, the technical jargon (such as compact SVD, orthonormal) is completely irrelevant to determining the smudged dimensions.

Solution:

①	⊗ k	○ m	○ n	○ r
②	○ k	○ m	⊗ n	○ r
③	○ k	○ m	○ n	⊗ r
④	⊗ k	○ m	○ n	○ r
⑤	○ k	○ m	○ n	⊗ r
⑥	○ k	○ m	○ n	⊗ r
⑦	○ k	○ m	○ n	⊗ r
⑧	○ k	⊗ m	○ n	○ r
⑨	○ k	○ m	⊗ n	○ r
⑩	○ k	○ m	○ n	⊗ r

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5. Matrix Inversion (10 points)

You landed your first job at 16A*tech* (the Bay Area's newest and hottest tech company), and your first assignment is to invert a matrix $A \in \mathbb{R}^{n \times n}$. You say "no problem", and implement Gaussian elimination. You obtain the following reduction of the augmented matrix:

$$[A|I] \longrightarrow [I|P].$$

The dimension n is extremely large, so the computation takes several days to complete, and you give your boss the matrix $P \in \mathbb{R}^{n \times n}$ just minutes before the deadline.

(a) (2 points) Your boss panics, saying "Oh, no! Your procedure only guarantees that $AP = I$ and not necessarily that $PA = I$." In *one sentence*, concisely explain why your boss thinks this might be an issue.

Concepts: This part tests your understanding of what the reduction $[A|I] \longrightarrow [I|P]$ means in terms of linear equations.

Solution: The reduction $[A|I] \rightarrow [I|P]$ corresponds to solving the system of equations $AX = I$ in variables $X \in \mathbb{R}^{n \times n}$, rather than the system of equations $XA = I$.

(b) (8 points) You try to calm them down, saying "Don't worry, the matrix also satisfies $PA = I$, and therefore P is the inverse of A just like you wanted. I'll prove it to you..."

Your proof consists of the following two steps (fill in the details as your answer to this question):

Step 1: Argue that your matrix P is the unique $Q \in \mathbb{R}^{n \times n}$ satisfying $AQ = I$.

Step 2: Prove that $PA = AP = I$. (Hint: consider the matrix $A(P + PA - I)$)

As suggested by part (a), you should not assume that A^{-1} exists. Proving that it does is the point of this problem.

Concepts: Step 1 asks you to interpret what the augmented matrix $[I|P]$ reveals about number of solutions to the corresponding system of linear equations. Step 2 requires you to use the distributive property of matrix multiplication together with the previously established property of P.

Solution:

Step 1: The reduction $[A|I] \rightarrow [I|P]$ implies there is a unique solution $X = P$ to the system of equations $AX = I$, since there are no free variables.

Step 2: Using the fact that $AP = I$, we follow the hint and evaluate

$$A(P + PA - I) = AP + APA - A = I + IA - A = I.$$

By the fact $AP = I$ and the uniqueness established in Step 1, we must have $P = P + PA - I$, which reduces to $PA = I$.

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6. Tomography (19 points)

Recall that in our simple tomography example of 4 pixels arranged into a 2×2 matrix, our initial set of measurements produced the following system of equations with unknowns $x_1, \ldots, x_4$ and measured intensities $b_1, \ldots, b_4$:

$$\begin{array}{lllll} x_1 & +x_2 & & & = b_1 \\ & & x_3 & +x_4 & = b_2 \\ x_1 & & +x_3 & & = b_3 \\ & x_2 & & +x_4 & = b_4 \end{array}$$

(a) (3 points) Write the above system of equations in matrix-vector form $A\vec{x} = \vec{b}$.

Concepts: Do you know how to write a system of equations in matrix-vector form?

Solution:

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}.$$

(b) (8 points) Use Gaussian elimination to find a basis for the nullspace of your matrix in part (a). Show your work.

Concepts: This question tests whether you know the definition of nullspace, and the mechanics of gaussian elimination for solving the system $A\vec{x} = \vec{0}$.

Solution:

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{array}\right]$$

$$\text{swap}(R_2, R_4) \quad \Rightarrow \quad \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right]$$

$$R_3 \leftarrow R_3 - R_1 + R_2 \quad \Rightarrow \quad \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right]$$

$$R_4 \leftarrow R_4 - R_3 \quad \Rightarrow \quad \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$R_1 \leftarrow R_1 - R_2 \quad \Rightarrow \quad \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

(Any valid sequence of elementary row operations are acceptable, provided you arrive at the correct matrix in rref)
So, solutions to $A\vec{x} = \vec{0}$ can be written as vectors $\vec{x}$ satisfying

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} x_4,$$

where x_4 is a free variable. Hence a basis for $N(A)$ is $[1 \ -1 \ -1 \ 1]^\top$.

(c) (2 points) Suppose $\vec{x}_0$ denotes the correct pixel values, which of course satisfy $A\vec{x}_0 = \vec{b}$. Give another solution $\vec{x}_1$ to the system of equations $A\vec{x} = \vec{b}$, satisfying $\vec{x}_1 \neq \vec{x}_0$. Leave your answer in terms of $\vec{x}_0$.

Concepts: This question checks whether you understand how nullspace relates to characterizing the set of solutions to a system of linear equations.

Solution: Another solution is $\vec{x}_0 + [1 \ -1 \ -1 \ 1]^\top$.

(d) (2 points) Suppose we add the measurement

$$x_1 + x_4 = b_5.$$

Will the resulting new system of equations always have a solution for any values $b_1, b_2, \ldots, b_5$? Completely fill in the circle next to the correct answer.

Concepts: This question checks whether you can determine consistency of a system of equations (e.g., by comparing dimension of the column space and dimension of $\vec{b}$).

Solution:

◯ Yes ⊗ No

The answer is no ($\vec{b}$ will be a vector in $\mathbb{R}^5$, but column-space of A has dimension at most 4, so we cannot guarantee a solution for any choice of $\vec{b}$).

(e) (4 points) Assuming a solution exists for the new system of equations in part (d), will the solution be unique? Justify your answer by showing work to support your conclusion.

Concepts: This question checks whether you understand how nullspace relates to uniqueness of a solution to a consistent system of linear equations.

Solution: Yes, the solution will be unique. One way of seeing this is to add the measurement to our (already reduced) system of equations and find it has trivial nullspace:

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{array}\right]$$

$$R_4 \leftarrow (R_4 - R_1)/2 \qquad \Rightarrow \qquad \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right].$$

We can see there will be no free variables, hence the system of equations with the new measurement has a unique solution (under the assumption of consistency).

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7. Dynamical Systems (26 points)

Define matrices $Q, R \in \mathbb{R}^{2\times 2}$ according to

$$Q = \begin{bmatrix} 0 & 3/4 \\ 1 & 1/4 \end{bmatrix}, \qquad R = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

(a) (5 points) Find the eigenvalues for the matrix Q.

Concepts: Do you know how to find the eigenvalues of a 2x2 matrix?

Solution: Note that

$$\det(Q - \lambda I) = (-\lambda)(1/4 - \lambda) - 3/4 = (\lambda - 1)(\lambda + 3/4).$$

So, the eigenvalues are $\lambda_1 = 1, \lambda_2 = -3/4$.

(b) (4 points) Consider a system with state vector $\vec{x}[n] \in \mathbb{R}^2$ at time $n \geq 1$ given by

$$\vec{x}[n] = Q\vec{x}[n-1].$$

Is there a non-zero vector $\vec{x}$ satisfying $\vec{x} = Q\vec{x}$? If yes, give one such vector.

Concepts: Can you find an eigenvector corresponding to a given eigenvalue (1 in this case)?

Solution: Yes, such a vector exists since the matrix has eigenvalue 1. To solve for it, we set up the system of equations $(Q - I)\vec{x} = 0$, which is explicitly written as

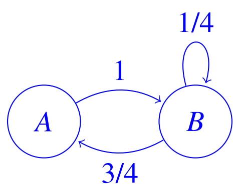
$$-x_1 + 3/4 x_2 = 0$$
$$x_1 - 3/4 x_2 = 0$$

One solution is $x_1 = 3/4, x_2 = 1$, giving the desired vector $\vec{x} = [3/4, \ 1]^\top$.

(c) (3 points) Draw the state-transition diagram for the system in part (b). Label your nodes "A" and "B".

Concepts: Do you know how to draw a state-transition diagram for a system of linear equations?

Solution:



(d) (4 points) Now, consider a system with state vector $\vec{w}[n] \in \mathbb{R}^2$ at time $n \geq 1$ given by:

$$\vec{w}[n] = \begin{cases} Q\vec{w}[n-1] & \text{if } n \text{ is odd} \\ R\vec{w}[n-1] & \text{if } n \text{ is even.} \end{cases}$$

Write expressions for $\vec{w}[1]$, $\vec{w}[2]$, $\vec{w}[3]$ and $\vec{w}[4]$ in terms of $\vec{w}[0]$ and Q and R. Write each answer in the form of a matrix-vector product.

Concepts: Given a description of a dynamical system, can you write out the state vectors at given time points in terms of the initial state and the transition matrices?

Solution:

$$\vec{w}[1] = Q\vec{w}[0], \quad \vec{w}[2] = RQ\vec{w}[0], \quad \vec{w}[3] = Q(RQ)\vec{w}[0], \quad \vec{w}[4] = (RQ)^2\vec{w}[0].$$

(e) (10 points) Suppose we start the system of part (d) with state $\vec{w}[0] = \begin{bmatrix} 11/14 & 3/14 \end{bmatrix}^\top$. Find expressions for $\vec{w}_{\text{even}}$ and $\vec{w}_{\text{odd}}$, which are defined according to

$$\vec{w}_{\text{even}} = \lim_{k\to\infty} \vec{w}[2k], \qquad \vec{w}_{\text{odd}} = \lim_{k\to\infty} \vec{w}[2k+1].$$

In words, $\vec{w}_{\text{even}}$ and $\vec{w}_{\text{odd}}$ describe the long-term behavior of the system at even and odd time-instants, respectively. (Hint: you can avoid computation by thinking about the system at even time-instants in terms of a state-transition diagram.)

Concepts: Following the hint, you should consider the dynamical system $\vec{w}[2k] = (RQ)^k \vec{w}[0]$ for $k \geq 1$. This is just like the dynamical systems you have considered previously, with transition matrix (RQ). If you compute this matrix product and draw the state diagram, you will find something that looks nearly identical to the page-rank example from lecture. So, this question tests whether you can recognize a familiar problem, perhaps presented in a slightly unfamiliar form (but guided by a hint).

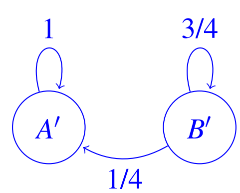
Solution: Following the hint, consider the system at even time-instants:

$$\vec{w}[2k] = (RQ)^k \vec{w}[0], \quad k \geq 0.$$

This looks like a dynamical system with transition matrix

$$RQ = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 3/4 \\ 1 & 1/4 \end{bmatrix} = \begin{bmatrix} 1 & 1/4 \\ 0 & 3/4 \end{bmatrix}.$$

The transition diagram for this system looks like:



This looks similar to the page rank example from lecture, where all traffic will end up on website A'. Hence, for the given choice of $\vec{w}[0]$ (whose entries add to one, and therefore can be thought of as representing fraction of traffic), we have

$$\vec{w}_{\text{even}} = \lim_{k\to\infty} \vec{w}[2k] = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \text{and, } \vec{w}_{\text{odd}} = Q\vec{w}_{\text{even}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

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8. Linearly Independent Solutions (5 points)

Let $A \in \mathbb{R}^{17\times 32}$ satisfy $\dim(C(A)) = 9$, where $C(A)$ denotes the column-space of A. How many linearly independent solutions can be found to the system of equations $A\vec{x} = \vec{0}$?

Note: Be careful. You are not being asked how many solutions exist for this system of equations, but rather how many *linearly independent solutions* can be found. You may just give a numerical answer; no work is required.

Concepts: Do you know (i) definition of dimension of a subspace (equal to max number of linearly independent vectors in a subspace); (ii) definition of null-space; and (iii) how dimension of null-space and column-space are related to matrix dimensions (i.e., rank nullity theorem)?

Solution: The number of linearly independent solutions is equal to the dimension of $N(A)$, which is the maximum number of linearly independent solutions to the equation $A\vec{x} = 0$, by definition. Hence, we use the rank-nullity theorem to compute:

$$\dim(N(A)) = 32 - \dim(C(A)) = 23.$$

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9. Inverses and Transposes (8 points)

Given an invertible matrix $A \in \mathbb{R}^{n\times n}$, use the definition of matrix inverse to prove that

$$(A^T)^{-1} = (A^{-1})^T.$$

Concepts: Do you know the definition of matrix inverse (i.e., $AA^{-1} = A^{-1}A = I$)? Do you remember what happens when you take transpose of a matrix product?

Solution: We are given that A is invertible, meaning that there is a matrix A^{-1} that satisfies

$$AA^{-1} = I \quad \text{and} \quad A^{-1}A = I.$$

We want something involving transposes, so the natural thing to do is take transpose of each of the above identities (using the fact you've seen: $(AB)^T = B^T A^T$) to obtain

$$(A^{-1})^T A^T = I \quad \text{and} \quad A^T (A^{-1})^T = I.$$

Hence, $(A^{-1})^T$ is equal to the inverse of A^T by definition of matrix inverse.

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10. Orthogonal Complements (16 points)

Consider the vector space $\mathbb{R}^n$, and let $\mathbb{U}$ be a subspace of $\mathbb{R}^n$. We define the set $\mathbb{U}^\perp \subset \mathbb{R}^n$, called the *orthogonal complement* of $\mathbb{U}$, according to

$$\mathbb{U}^\perp = \{\vec{x} \in \mathbb{R}^n \mid \vec{u}^\top \vec{x} = 0 \text{ for all } \vec{u} \in \mathbb{U}\}.$$

(a) (4 points) Show that $\mathbb{U}^\perp$ is a subspace of $\mathbb{R}^n$.

Concepts: Do you know the definition of a subspace, and can you verify it on a given example?

Solution: We should show that $\mathbb{U}^\perp$ is closed under scalar multiplication and vector addition. To this end, let $\vec{x}_1, \vec{x}_2 \in \mathbb{U}^\perp$ and $\alpha, \beta \in \mathbb{R}$. Then, for all $\vec{u} \in \mathbb{U}$, we have

$$(\alpha \vec{x}_1 + \beta \vec{x}_2)^T \vec{u} = \alpha \vec{x}_1^T \vec{u} + \beta \vec{x}_2^T \vec{u} = 0,$$

where the last identity follows by definition of $\vec{x}_1, \vec{x}_2 \in \mathbb{U}^\perp$. Hence, $(\alpha \vec{x}_1 + \beta \vec{x}_2) \in \mathbb{U}^\perp$, and therefore $\mathbb{U}^\perp$ is a subspace.

(b) (4 points) Find a concise expression for the intersection $\mathbb{U} \cap \mathbb{U}^\perp$. Justify your answer.

Concepts: You saw the operation $\cap$ for subspaces in your homework; can you use definitions to compute it for a specific example?

Solution: If $\vec{x} \in \mathbb{U}^\perp$, then $\vec{x}^T \vec{u} = 0$ for any choice of $\vec{u} \in \mathbb{U}$. In particular, if $\vec{x} \in \mathbb{U}^\perp \cap \mathbb{U}$, then we must have

$$0 = \vec{x}^T \vec{x} = \sum_{i=1}^{n} x_i^2 \quad \Leftrightarrow \quad x_i = 0 \text{ for all } i \quad \Leftrightarrow \quad \vec{x} = \vec{0}.$$

Hence, $\mathbb{U}^\perp \cap \mathbb{U} = \{\vec{0}\}$.

(c) (6 points) Working in dimension $n = 3$, consider the subspace

$$\mathbb{U} = \text{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

Find a basis for $\mathbb{U}^\perp$.

Concepts: Can you formulate the problem of computing $\mathbb{U}^\perp$ as a system of linear equations and solve? This is almost identical to how we compute the nullspace of a matrix: we formulate an appropriate system of linear equations, and then solve.

Solution: To characterize $\mathbb{U}^\perp$, we should find the set of vectors $\vec{x}$ such that $[1,2,3]\vec{x} = 0$ and $[0,1,1]\vec{x} = 0$. This can be done, for example, by reducing the augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \quad \Rightarrow \quad \text{solutions are of form: } \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} x_3.$$

Hence, the vector $[-1,-1,1]^T$ is a basis for $\mathbb{U}^\perp$.

(d) (2 points) For the subspaces $\mathbb{U}$ and $\mathbb{U}^\perp$ of part (c), show that $\mathbb{U} + \mathbb{U}^\perp = \mathbb{R}^3$.

Concepts: Do you know that three linearly independent vectors in $\mathbb{R}^3$ will span $\mathbb{R}^3$?

Solution: We have

$$\mathbb{U} + \mathbb{U}^{\perp} = \text{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right\} = \mathbb{R}^3,$$

the latter identity follows since the three vectors are linearly independent (this actually follows from part (b)!), and therefore form a basis for $\mathbb{R}^3$.