EECS 16A Designing Information Devices and Systems I
Spring 2021 Final Exam – Instructions

Read the following instructions before the exam.

Good luck on the final exam! You've studied hard and we are rooting for you to do well!

Our advice to you: if you can't solve a particular problem, move on to another, or state and solve a simpler one that captures at least some of its essence. You will perhaps find yourself on a path to the solution. **We believe in you!**

Format & How to Submit Answers

In this exam there are 10 problems (2 introductory questions and 8 exam questions containing subparts) with varying point number. The problems are of varying difficulty, so pace yourself accordingly and avoid spending too much time on any one question until you have gotten all of the other points you can. If you are having trouble with one problem, there may be easier points available later in the exam!

Complete your exam using either the template provided or appropriately created sheets of paper. Either way, you should submit your answers to the *Gradescope* assignment that is marked Final for your specific exam group. Make sure you submit your assignment to the correct *Gradescope* assignment. You MUST select pages for each question. We cannot grade your exam if you do not select pages for each question. If you are having technical difficulties submitting your exam, you can email your answers to `eecs16a@berkeley.edu`.

In general, show all your work legibly to receive full credit; we cannot grade anything that we cannot read. For some problems, we may try to award partial credit for substantial progress on a problem, and showing your work clearly and legibly will help us do that.

Timing & Academic Honesty

You are expected to follow the rules provided in the Exam Proctoring Guidelines.
https://docs.google.com/document/d/10pnWwxyZ40nlpbCM4aOYTxXOjc36sIQaMx9m8zyaR8w/edit?usp=sharing
The exam will be available to you at the link sent to you via email. The exam will start at 11:30am Pacific Time, Wednesday, May 12th, 2021, unless you have an exam accommodation. If you experience technical difficulties and cannot access your exam, let us know by making a private post on Piazza and we will try to help.

You have 180 minutes (3 hours) for the exam, with 40 minutes of extra time for scanning and submitting to *Gradescope*. Most of you will have to submit your exam by 3:10pm unless you have another accommodation. Late submissions will be penalized exponentially. An exam that is submitted N minutes after the end of the submission period will lose 2^N points. This means that if you are 1 minute late you will lose 2 points; if you are 5 minutes late you will lose 32 points and so on.

This is a closed-note, closed-book, closed-internet, and closed-collaboration exam. Calculators are not allowed. You may consult **three** handwritten 8.5" by 11" cheat sheets (front and back of three pieces of paper). Do not attempt to cheat in any way. We have a zero tolerance policy for violations of the Berkeley Honor Code.

EECS 16A Designing Information Devices and Systems I
Spring 2021 Final Exam

1. HONOR CODE

If you have not already done so, please copy the following statements into the box provided for the honor code on your answer sheet, and sign your name.

I will respect my classmates and the integrity of this exam by following this honor code. I affirm:

- *I have read the instructions for this exam. I understand them and will follow them.*
- *All of the work submitted here is my original work.*
- *I did not reference any sources other than my allocated reference cheat sheet(s).*
- *I did not collaborate with any other human being on this exam.*

2. (a) **(2 Points) Name someone who makes you feel happy.**
All answers will be awarded full credit.

(b) **(2 Points) What are you looking forward to over the summer?**
All answers will be awarded full credit.

3. (22 points) Linear Algebra

(a) (4 points) Use Gaussian Elimination to determine if the following system of equations has either **no solution, one solution, or infinite solutions**. If there is a single solution please write it explicitly, and if there are infinite solutions please **specify the full set of solutions**.

$$\begin{aligned} x+y+z &= 3 \\ 4x+3y+2z &= 5 \\ 7x+3y+4z &= 8 \end{aligned}$$

(b) (4 points) **Find the null space $N(\mathbf{A})$ of the following matrix.**

$$\mathbf{A} = \begin{bmatrix} -6 & 8 & 1 \\ 3 & -1 & 1 \end{bmatrix}$$

(c) (6 points) Ashwin has lost his op-amp! Let its location in 2D be denoted by the vector $\vec{x} \in \mathbb{R}^2$. **Set up a set of *linear* equations in the form $\mathbf{A}\vec{x} = \vec{b}$ and solve for $\vec{x}$** based on the provided information:

i. It is 2 units away from $(1,3)$.
ii. It is $\sqrt{10}$ units away from $(2,4)$.
iii. It is 3 units away from $(-2,1)$.

(d) (4 points) For this part you will need to sketch vectors on a 2D plane. Make sure your plot clearly labels the x and y axes.

i. **Plot the vector** $\vec{v} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$. Label the vector and clearly indicate the vector components on the plot.
ii. On the same graph **plot the null space of the matrix $\mathbf{A} = \begin{bmatrix} -3 & 2 \end{bmatrix}$**.

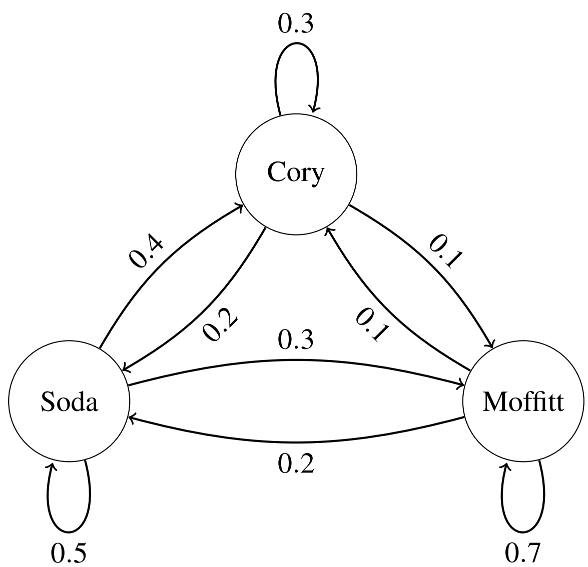
(e) (4 points) Let $\vec{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$. **Compute the following inner product**:

$$\left\langle 2\vec{x} + \vec{y}, \frac{1}{2}\vec{y} \right\rangle$$

4. (16 points) Transitioning Back to Campus

UC Berkeley administrators are drafting up the social distancing guidelines for reintroducing engineering students back onto campus. They know these students typically spend a lot of time in Soda Hall, Cory Hall, and Moffitt Library and they need to determine how many students can be re-introduced without violating any Covid-related building occupant capacities. They have an initial transition-state model acquired from prior years, but need your help.

(a) (6 points) Prior on-campus student traffic data lead administrators to assemble the following transition diagram describing how students move between these buildings (where each arrow represents the proportion of students moving).



The current number of students in each building at time-step t is given by the state vector $\vec{x}[t]$ defined as:

$$\vec{x}[t] = \begin{bmatrix} x_C[t] \\ x_S[t] \\ x_M[t] \end{bmatrix} = \begin{bmatrix} \text{number of students in Cory at time } t \\ \text{number of students in Soda at time } t \\ \text{number of students in Moffitt at time } t \end{bmatrix}$$

i. **Explicitly write out the transition matrix T from the provided diagram** such that $\mathbf{T}\,\vec{x}[t] = \vec{x}[t+1]$.

ii. Does this model account for all students leaving or staying at each of these three buildings? In other words, **is the system conservative?** Justify your answer.

(b) (10 points) Berkeley administrators have just modified the original transition matrix model based on certain courses/labs remaining in remote operation, hence the new transition matrix

$$\mathbf{M} = \begin{bmatrix} 0.6 & 0.0 & 0.0 \\ 0.2 & 0.4 & 0.6 \\ 0.2 & 0.6 & 0.4 \end{bmatrix}.$$

State guidelines impose limits on the typical number of students occupying each building. To predict if the number of students in each building will meet guidelines, you decide to examine the steady-state behavior of the transition system.

i. You are given that: $\lambda_1 = 1$, $\lambda_2 = -0.2$, and $\lambda_3 = 0.6$.
Identify a steady state vector $\vec{x}_{steady}$ such that $\mathbf{M}\vec{x}_{steady}$ = $\vec{x}_{steady}$.

ii. State guidelines impose the following limits on the number of students occupying Cory, Soda, and Moffitt:

$$\vec{x}_{limit} = \begin{bmatrix} 100 \\ 60 \\ 80 \end{bmatrix}.$$

It is also anticipated that the following number of students will be in each building at the start of the day $\vec{x}_0 = [20, 50, 70]^T$. **Argue whether or not the state guidelines $\vec{x}_{limit}$ will be satisfied in the steady state (after an infinite number of time-steps occur).**

5. (16 points) Negative Resistance Circuit

While waiting for lab checkoff you decide to fiddle with some op-amp topologies, and stumble upon the circuit below (Figure 1) that behaves like a negative-valued resistor! In this question you will be guided through a method for finding the equivalent resistance of the circuit. Afterwards we will investigate one potential application of this circuit.

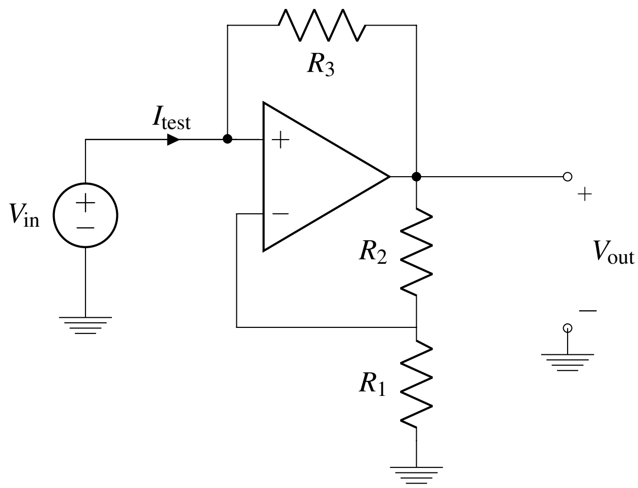


Figure 1: Op-amp circuit that behaves like a negative-valued resistor.
I_{test} is defined as the current from the voltage source V_{in} towards the u_+ input op-amp node.

(a) (6 points)

i. **Find an equation for V_{out} in terms of V_{in}, I_{test}, and R_3.**

ii. **Find an equation for V_{out} in terms of V_{in}, R_1, and R_2.**

iii. The equivalent resistance looking into the circuit will be the voltage at the node divided by the current going into the node: $R_{\text{eq}} = \frac{V_{\text{in}}}{I_{\text{test}}}$. **What is the equivalent resistance of this circuit?** You should use the results from parts (i) and (ii), and your answer should not contain V_{out}.

(b) (4 points) In lab you are using a current source to test a load resistance, but you find that the load current I_L depends on the load resistance R_L. You infer that the current source has an internal source resistance R_0 and come up with the following model (Figure 2) for your circuit.

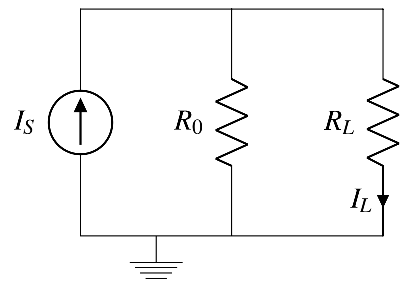


Figure 2: Model diagram for a current source with internal resistance R_0.

What is I_L in terms of given variables?
Your solution should be in terms of R_0, R_L, and I_S.

(c) (6 points) You decide to use the op-amp circuit from part (a) in order to make the load current in part (b) independent of R_L. The circuit from part (a) can be modeled as a resistor with resistance $R_a < 0$ (a negative value). This new circuit element R_a will be wired in parallel with the current source as shown in Figure 3.

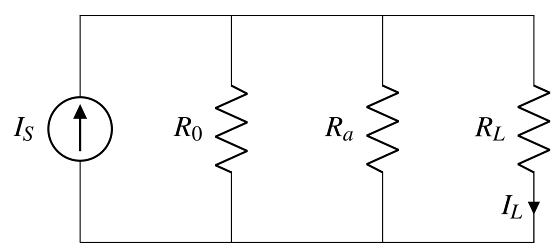


Figure 3: Diagram for the application of the *negative resistance* circuit to a current source.

Choose a value for $R_\mathbf{a}$ such that the current through the load resistors I_L is equal to the current through the source current I_S. Show that $I_L = I_S$ with your chosen R_a (i.e. do not just guess a value).

6. (18 points) Designing a light meter

Our plants keep dying from not getting enough sun! To prevent this we want to design a circuit to measure the light the plant gets. We will start with a photodetector, which we can model as a current source I_s. When the plants are getting sufficient sun exposure, the current source outputs 5 nA $= 5 \times 10^{-9}$A. Conversely, when they are *not* getting enough sun exposure the current source outputs 0 nA.

(a) (6 points) We wire up the current source I_s into the capacitor circuit shown in Figure 4 below.

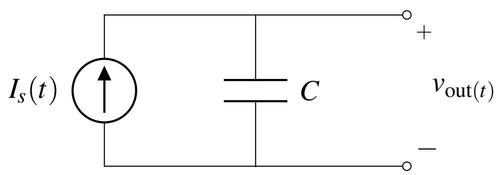


Figure 4: Light meter circuit, where the current source $I_s(t)$ models the photodetector.

Find an expression for $v_{out}(t)$ in terms of $I_s(t)$, C, and t when under *constant* light exposure ($I_s(t) = 5$ nA). Then identify the capacitor value C such that, after 1 hour under exposure, the capacitor voltage is $V_{out} = 5V$. Assume the initial voltage on the capacitor is 0V.

(b) (6 points) We would like to use the previous circuit to power a separate LED device that indicates the state of sun exposure on our plant. The device indicates sufficient sun exposure when +5V is applied across it, and conversely indicates the plant is critically underexposed when -5V is applied across it. **Design a circuit *using a comparator* that outputs +5V once the plant has received at least 1 hour of full exposure, and otherwise outputs -5V.** You have a voltage source $V_a(t)$ which corresponds to $V_{out}(t)$ from part (a).

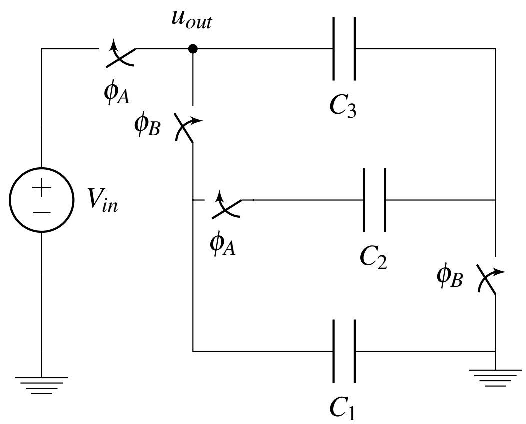
Regardless of your answer in part (a) assume $V_a(t)$ functions exactly as previously described; so $V_a(t) = 5V$ after an hour of full exposure. You may use as many voltage sources as you would like. Label the comparator rails and any voltage sources you include (with explicit voltage values).

(c) (6 points) We want to use the comparator output from the previous circuit to talk to a microcontroller. However, your microcontroller can only read 0V to 5V, instead of the -5V to 5V output voltage from the comparator in part (b). Furthermore, you do not have access to any other comparator. **Design a circuit, without using a comparator, that scales and shifts an input voltage in the range** $-5V \leq V_{in} \leq +5V$ **to produce an output voltage** $0V \leq V_{out} \leq +5V$**.** Use the voltage source $V_{in} = V_b(t)$ to model the output of part (b). **You are limited to only use circuit elements provided with your lab**, which entails 4 resistors, two op-amps, and one constant voltage source. For any resistors or voltage sources that you use in your design, you may pick any component value, but please clearly label and specify its value.

Hint: There are multiple possible solutions to this sub-part.

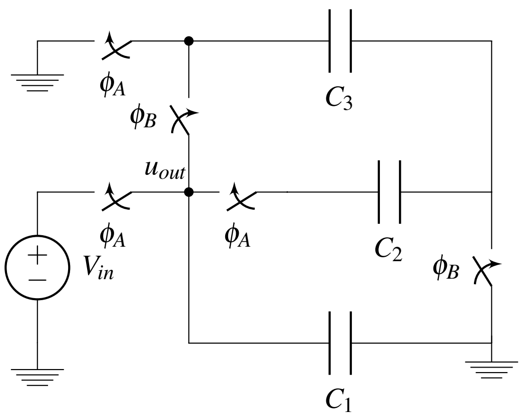
7. (16 points) Charge sharing check-in

(a) (8 points) Let us analyze the capacitor circuit shown below. Let us set all of the capacitors to have the same capacitance $C_1 = C_2 = C_3 = C$. Assume that all capacitors start without any initial charge, i.e. they are completely discharged before phase A.



The switches for phase A close first and the capacitors charge up completely. Those switches are then disconnected and the switches for phase B are closed. **What is the voltage at node u_{out} in phase B in terms of C and V_{in}?** Draw out the two phases of the circuit for partial credit.

(b) (8 points) In this part we will analyze a slightly different circuit, shown below. Let us again set all of the capacitors to have the same capacitance $C_1 = C_2 = C_3 = C$. Assume that all capacitors start without any initial charge, i.e. they are completely discharged before phase A. The switches for phase A close first and the capacitors charge up completely. Those switches are then disconnected and the switches for phase B are closed.



What is the voltage at u_{out} in phase B in terms of C and V_{in}? Draw out the two phases of the circuit for partial credit.

8. (16 points) Transatlantic Telegraph Cable

The year is 1956, and secret agents Alice & Bob have been deployed to New York and London respectively. Alice regularly needs to send Bob sensitive information, and they just caught word of a recently established transatlantic telegraph cable TAT-1 between the continents, which has a much faster communication speed than mailing letters.

(a) (4 points) Alice and Bob use conventional mail to agree on a specific binary code for conveying their secret messages. Alice found a 6-element code in one of her old training manuals, but unfortunately one of the numbers in code is illegible:

$$\vec{s}[n] = [\,+1\,,\,-1\,,\,+1\,,\,+1\,,\,-1\,,\,\gamma\,]$$

(where γ represents the missing entry, which will be either +1 or -1). Luckily her manual also provides a diagram of this code's auto-correlation function, plotted in Figure 5 below.

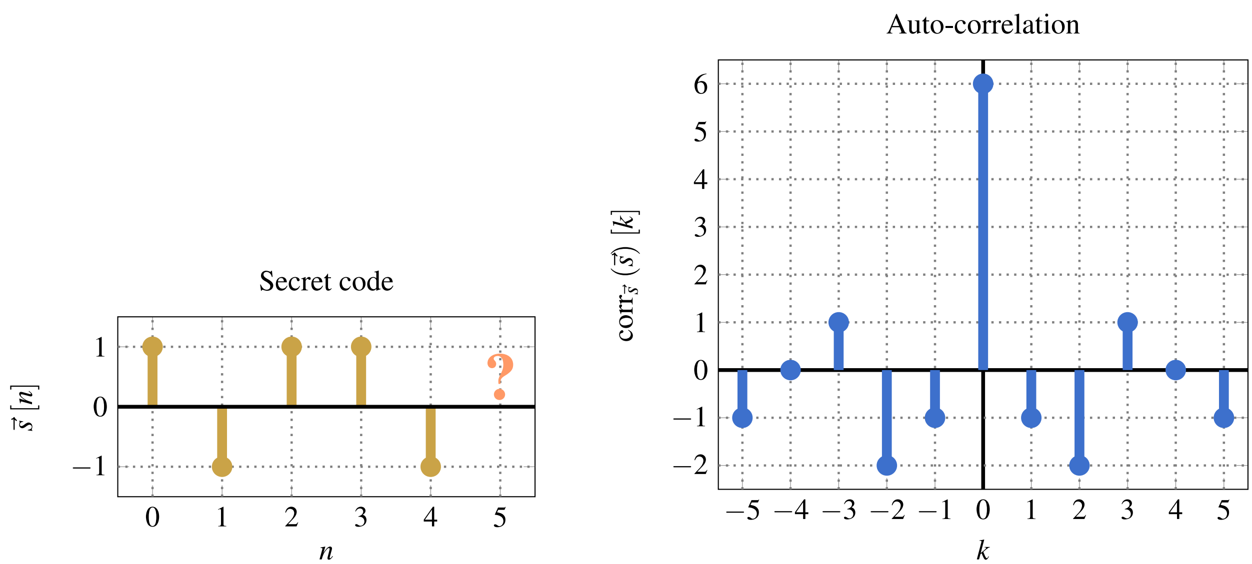


Figure 5: Alice's secret code (left) has a length of 6 elements but the last entry $\vec{s}[5] = \gamma$ is unknown. The auto-correlation (right) $\text{corr}_{\vec{s}}(\vec{s})[k]$ of the secret code.

Identify the missing entry γ (either $+1$ or -1) of the code from Alice's manual. Provide explicit reasoning to justify your answer.

(b) (4 points) Unfortunately, enemy counter-intelligence managed to intercept Alice's mail to Bob, so she decides to use a new secret code (shown in Fig. 6):

$$\vec{s}[n] = [\ 1\ ,\ -1\ ,\ 1\ ,\ -1\ ,\ -1\ ,\ -1]$$

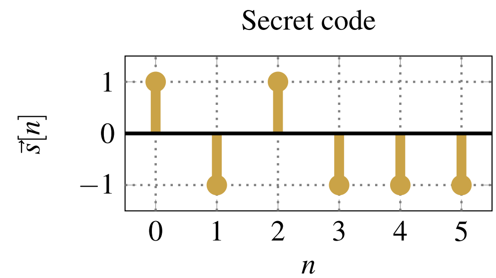


Figure 6: Alice's newest secret code (top).

In order to communicate through the transatlantic cable, Alice and Bob need to determine the transmission delay between them. This will help Bob find the starting time stamp of Alice's messages. Alice starts transmitting the secret code repeatedly. She transmits the first element of each 6-element code at time stamps $n = [-12, -6, 0, +6, +12]$. At the receiving side, Bob sees the code but with some delay. Bob's best guesses of the delay are $3T$, $4T$, or $5T$, where T denotes the time interval between adjacent time stamps.

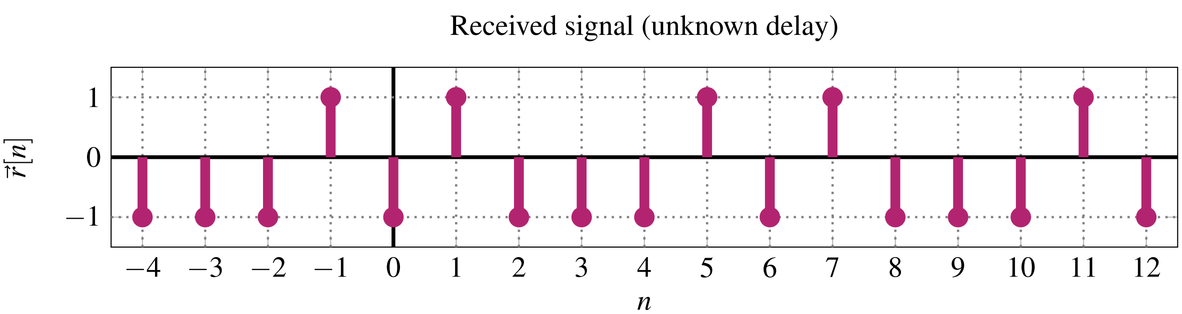


Figure 7: Bob's received signal, shifted by some unknown delay (bottom)

Bob receives the signal $\vec{r}[n]$ shown in the lower Figure 7 above. **Based on Bob's guesses, what is the actual time delay in terms of T? Justify mathematically.** You can assume that the delay is an integer number of T (i.e. no decimals).

(c) (4 points) Now that Alice and Bob have determined the delay, Alice can send messages using the secret code. Alice's messages contain binary symbols (composed ± 1 values), and she encodes her messages by multiplying each of her message symbols with the 6-element secret code. So if she sends a message with 18 elements, it must contain $18/6 = 3$ symbols of information. For this part assume there is no time delay between transmission and reception.

Alice sends a 18-element-long signal to Bob. However, some noise corrupts the signal, so the signal Bob receives $\vec{r}[n]$ contains imperfect samples instead of +1 and -1. He decides to take the cross correlation with the secret code to try to decode the message.

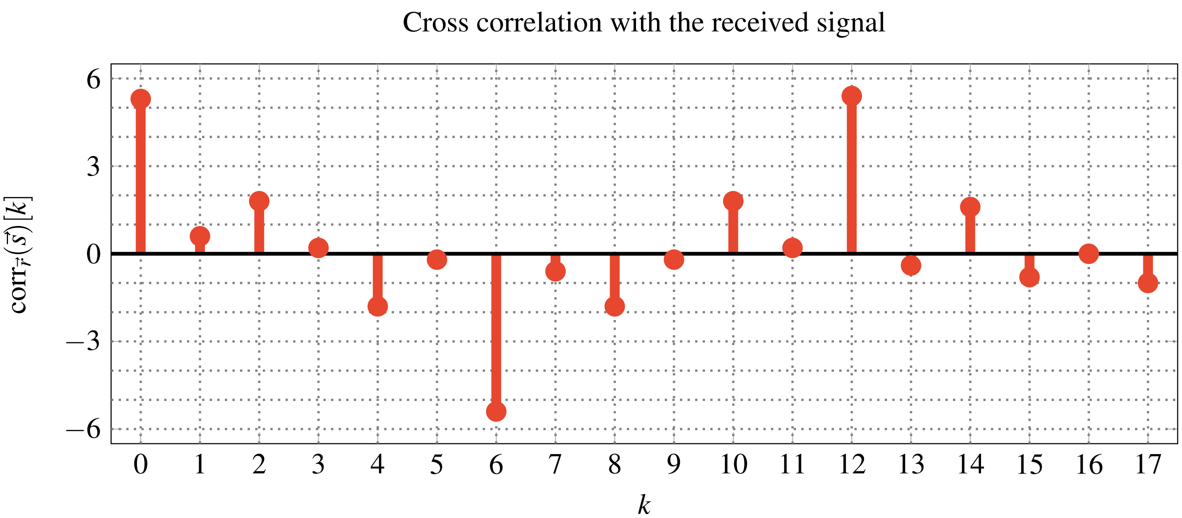


Figure 8: Cross correlation of the secret code with Bob's received signal, containing 3 symbols of information. $k < 0$ and $k > 17$ are ignored.

The cross correlation with the secret code for shifts $k = 0$ through $k = 17$ is:

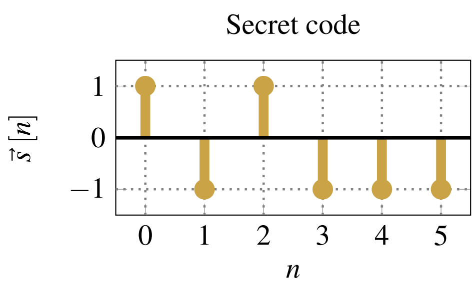
$\text{corr}_{\vec{r}}(\vec{s})[k] = [5.3, 0.6, 1.8, 0.2, -1.8, -0.2, -5.4, -0.6, -1.8, -0.2, 1.8, 0.2, 5.4, -0.4, 1.6, -0.8, 0, -1]$

Based on the cross-correlation with the received waveform, extract the message sent to Bob. Justify your answer.

(d) (4 points) Enemy forces have cut the cable to prevent Alice & Bob from communicating. Alice can go underwater to fix the break, but she must first identify the break location in the Atlantic ocean. She expects the broken point to echo her signal back along the cable, i.e. if she transmits her signal, it will reflect at the break and she will receive the signal back after some delay. She decides to transmit her 6-bit code

$$\vec{s}[n] = [\ 1\ ,\ -1\ ,\ 1\ ,\ -1\ ,\ -1\ ,\ -1\ ]$$

and monitor the signal $\vec{r}[n]$ that echoes back. She computes the cross-correlation of the echo with the secret code, i.e. $\text{corr}_{\vec{r}}(\vec{s})[k]$, shown in Figure 9.



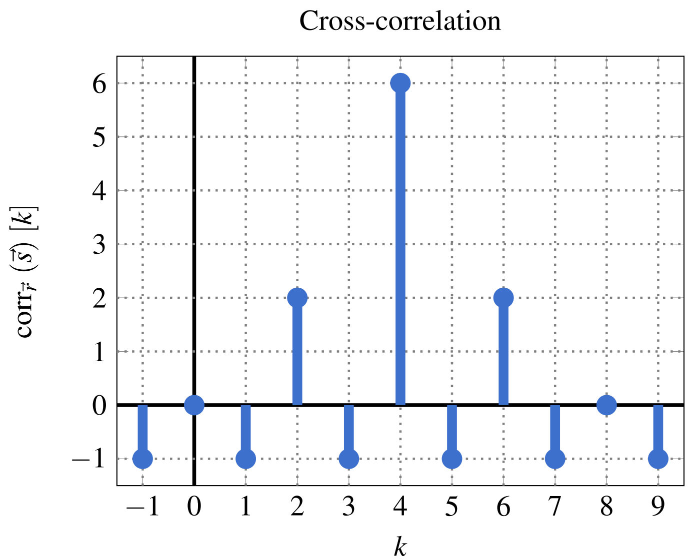


Figure 9: 6-bit secret code (left).
Cross-correlation (right) $\text{corr}_{\vec{r}}(\vec{s})[k]$ of the echo with the secret code.

Assume the time interval between adjacent time stamps is $T = 1$ ms $= 10^{-3}$ s, and that the signal travels in the telegraph cable at a speed of $v = 2 \times 10^8$ m/s (the speed of light is $c \approx 3 \times 10^8$ m/s in a vacuum, but in the cable medium it is $v = \frac{2}{3}c$). **How far is the broken point from Alice's location?**

HINT: Remember to account for the fact that the received signal has taken a full round trip, which is double the distance from Alice to the break.

9. (16 points) Cool Predictions

You have just been contracted by PG&E to predict daily energy use by the UC Berkeley campus based on local weather conditions. They have provided data of last year's daily energy usage along with corresponding weather reports.

(a) (5 points) You hypothesize a linear model based on the phase of the moon (represented as an integer between 0 and 8) and the season (represented as an integer between 0 and 3).

$$E \;=\; \alpha_P\, x_P \;+\; \alpha_S\, x_S \tag{1}$$

where E is the daily energy usage (in kilowatt-hours), x_P represents the moon phase, and x_S corresponds to the seasons. To get an initial approximation for the model parameters α_P and α_S, you sample data from 3 days to set up a linear system; the data have been printed in Table 1 below.

$\mathbf{x_P}$	$\mathbf{x_S}$	**E**
1	0	4
0	2	2
1	1	5

Table 1: Campus daily energy-usage
(x_P is the moon phase, x_S is the season, and E is the energy usage)

Explicitly set up the linear system of equations $\mathbf{D}\vec{a} = \vec{E}$, then from this system compute the least-squares solutions for $\hat{\vec{a}} = \begin{bmatrix} \alpha_P \\ \alpha_S \end{bmatrix}$.

(b) (4 points) After speaking with the on-campus facility experts, you realize that a majority of energy is actually used for heating/air conditioning for indoor facilities (affected by outdoor temperature) and the outdoor pools (affected by wind conditions). This guides you to a new prediction model

$$E \ = \ \alpha_T (x_T - 15)^2 + \alpha_W \ x_W + \alpha_R \tag{2}$$

where E is daily energy usage (in kilowatt-hours), x_T corresponds to temperature (in Celsius), x_W corresponds to wind speed (in meters per second), and α_R accounts for all non-heating-related energy usage. Now you would like to use data from last year's logs to identify the least-squares solution for the parameters of this new model $\vec{a} = \begin{bmatrix} \alpha_T \\ \alpha_W \\ \alpha_R \end{bmatrix}$.

$\mathbf{x_T}$ (°C)	$\mathbf{x_W}$ (m/s)	E (kWh)
25	4	300
20	5	220
10	1	250
15	5	280
12	9	350

Table 2: Selected energy/weather data logs.
(x_T is the temperature, x_W is the wind speeds, and E is the energy usage.)

Provided the data in Table 2 above, explicitly set up the linear system $\mathbf{D}\vec{a} = \vec{E}$ needed to solve for the least-squares solution of the parameter vector $\vec{a}$. This means you must write out all of the matrix $\mathbf{D}$ and vector $\vec{E}$ elements for full credit.

Note: you are not asked to actually find the least-squares solutions for this part!

(c) (3 points) Now that you spent some time playing around with different prediction models, you have finally developed a surprisingly simple model (with only 3 parameters) that accurately fits last year's the energy and weather data.

Your next task is to test the model against this year's weather and energy data. Unfortunately the latest data (matrix **D** below) keeps causing your code to crash! Checking the data, you try tweaking a single entry (denoted by η) in **D** and re-running. Lo and behold... it works!

$$\mathbf{D} = \begin{bmatrix} 30 & 25 & 5 \\ 25 & 21 & 4 \\ 10 & 6 & 4 \\ 10 & \eta & 5 \\ 20 & 15 & 5 \end{bmatrix}$$

The original error message said "Math error: Solution cannot be computed." **Explain what is likely causing this error. Then use this reasoning to identify the original value of the data entry η that causes the code to crash.**

(d) (4 points) As a final test, you apply all 3 of your developed models (from question parts a, b, c) to predict daily energy usage over the last 3 days from the most recent weather data. These models generate the following daily energy usage estimates $\hat{\vec{E}}_a$, $\hat{\vec{E}}_b$, and $\hat{\vec{E}}_c$, shown below along with the actual energy values $\vec{E}$.

$$\vec{E} = \begin{bmatrix} 1 \\ 3 \\ 0.5 \end{bmatrix} \qquad \hat{\vec{E}}_a = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \hat{\vec{E}}_b = \begin{bmatrix} 0 \\ 2 \\ 1.5 \end{bmatrix} \quad \hat{\vec{E}}_c = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$$

Compute the squared error for each of the models. Based on that, which model best agrees with the actual daily usage data?

10. (16 points) Symmetric and PSD Matrices

The eigenvectors corresponding to distinct eigenvalues of a general matrix $\mathbf{A}$ are linearly independent, and the eigenvalues of said matrix can be any real (or even complex!) numbers. In this question we consider two special classes of matrices (used ubiquitously in machine learning) and prove some essential properties about their eigenvalues/vectors.

(a) (8 points) A *symmetric* matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ is a square matrix such that $\mathbf{A} = \mathbf{A}^T$. Let $\vec{u}$ and $\vec{v}$ be eigenvectors of symmetric matrix $\mathbf{A}$ with distinct eigenvalues λ and μ respectively. **Show that the eigenvectors $\vec{u}$ and $\vec{v}$ are *orthogonal*.**

Hint: Consider the expression $\vec{v}^T \mathbf{A} \vec{u}$.

(b) (8 points) A symmetric matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ is called *positive semi-definite* if $\vec{x}^T \mathbf{A} \vec{x} \geq 0$ for any $\vec{x} \in \mathbb{R}^n$. Assume that the eigenvalues of any symmetric matrix $\mathbf{A}$ are real. **Show that a symmetric positive semi-definite matrix A has all non-negative eigenvalues (i.e, $\lambda \geq 0$).**

Hint: Apply definitions of eigenvectors, eigenvalues, and positive semi-definiteness. You will NOT need to use the fact that $\mathbf{A}$ is symmetric; this just ensures that the eigenvalues of $\mathbf{A}$ are real, which you do not need to prove.