

Final Solution

General Notes

- This exam has a combination of multiple choice questions and fill in the blank.
- This exam will be partially autograded. **You must adhere to the following format to receive full credit:**
 - For fill in the blank questions, **legibly write your final answer entirely in the provided boxes.**
 - For questions with **circular bubbles**, select exactly *one* choice, by filling the bubble ●.
 - You must choose either this option. ○ Or this one, but not both!
 - For questions with **square boxes**, you may select *multiple* choices, by filling the squares ■.
 - You could select this choice. You could select this one too!

1. HONOR CODE

Please read the following statements of the honor code, and sign your name (you don't need to copy it).

I will respect my classmates and the integrity of this exam by following this honor code. I affirm:

- *I have read the instructions for this exam. I understand them and will follow them.*
- *All of the work submitted here is my original work.*
- *I did not reference any sources other than my unlimited printed resources.*
- *I did not collaborate with any other human being on this exam.*

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Tell us about something you are looking forward to this summer! (1 point) *All answers will be awarded full credit.*

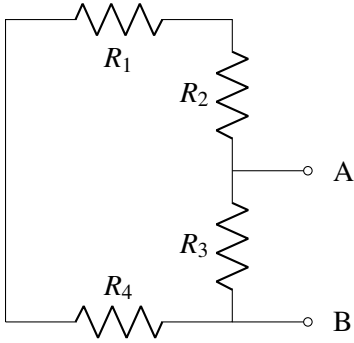
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2. Circuit Networks (3 points)

For each part, find the equivalent resistance or capacitance between terminals A and B.

You may use the \parallel operator in your final expressions.

(a) (1 point)

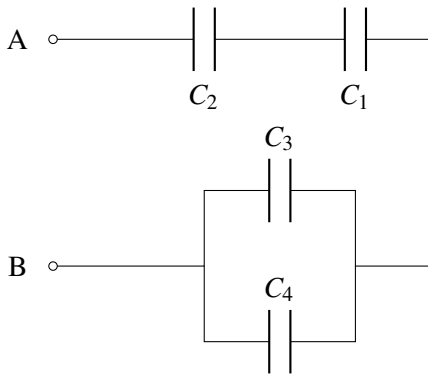


$R_{eq} =$

Solution:

$$R_{eq} = R_3 \parallel (R_1 + R_2 + R_4)$$

(b) (1 point)

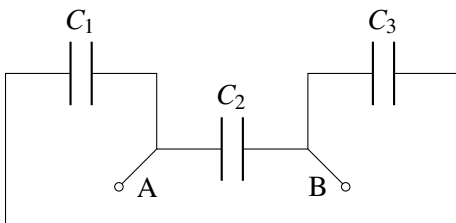


$C_{eq} =$

Solution:

$$C_{eq} = C_1 \parallel C_2 \parallel (C_3 + C_4)$$

(c) (1 point)



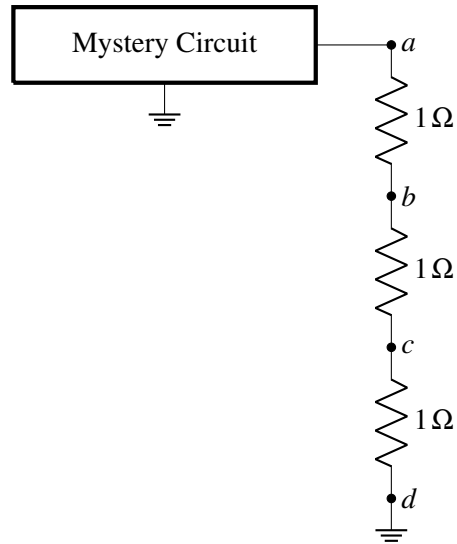
$C_{eq} =$

Solution:

$$C_{eq} = C_2 + (C_1 \parallel C_3)$$

3. Mystery Circuit Voltage Divider (6 points)

You are given a “mystery circuit” in a box with a part sticking out, as in the diagram below.



(a) (2 points) The voltage at node *a* is measured to be 6 V. Find the voltages at nodes *b* and *c*.

Node *b* voltage: V

Node *c* voltage: V

Solution:

We can use the voltage divider equation twice to find the node voltages

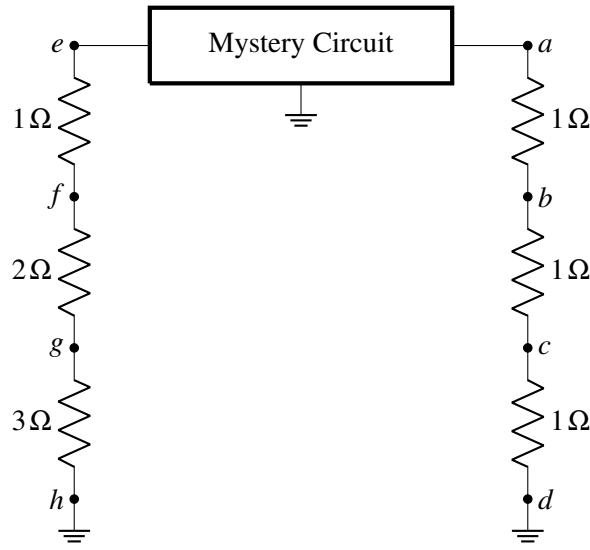
At node *b*:

$$u_b = \frac{1\ \Omega + 1\ \Omega}{1\ \Omega + 1\ \Omega + 1\ \Omega} 6\text{V} = \frac{2}{3} \cdot 6\text{V} = 4\text{V}$$

And at node *c*:

$$u_c = \frac{1\ \Omega}{1\ \Omega + 1\ \Omega + 1\ \Omega} 6\text{V} = \frac{1}{3} \cdot 6\text{V} = 2\text{V}$$

- (b) (2 points) Imagine there's another part of the circuit added on the left, with differently valued resistors as shown.



Given that both node a and node e are measured to be 6 V , compare the node voltages below.

Use either: $>$, $=$, or $<$.

i. e a

iii. g c

ii. f b

iv. h d

Solution: Using the results from part (a) and reusing the voltage divider formula for nodes f and g :
At node f :

$$u_f = \frac{2\Omega + 3\Omega}{1\Omega + 2\Omega + 3\Omega} 6\text{ V} = \frac{5}{6} \cdot 6\text{ V} = 5\text{ V}$$

And at node g :

$$u_g = \frac{3\Omega}{1\Omega + 2\Omega + 3\Omega} 6\text{ V} = \frac{3}{6} \cdot 6\text{ V} = 3\text{ V}$$

The node voltages at h and d are both zero since they are both tied to ground.

Then comparisons between node voltages are

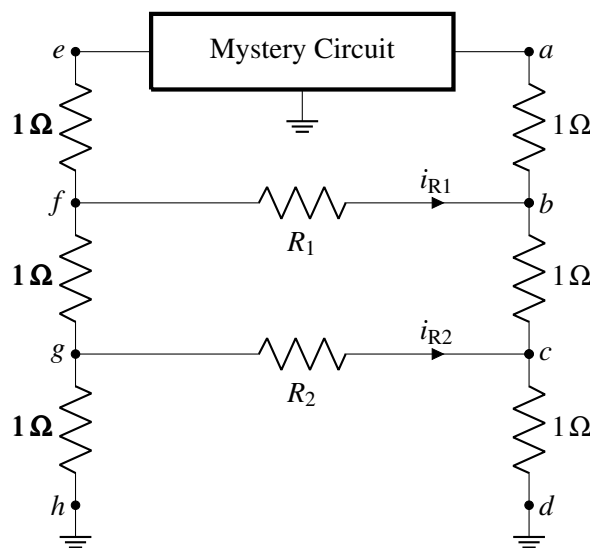
i. e a

iii. g c

ii. f b

iv. h d

- (c) (2 points) Now the resistors on the left side are all adjusted to be equal to $1\ \Omega$, and an additional resistor $R_1 = 0.5\ \Omega$ is connected between nodes b and f , and resistor $R_2 = 2\ \Omega$ is connected between nodes c and g . You again measure the voltage at both node a and node e to be $6\ \text{V}$.



- i. What is the current i_{R1} through resistor R_1 ?

$$i_{R1} = \boxed{} \text{ A}$$

Solution:

Since the node voltages at e and a are equal and the resistive ladder on the left matches the resistive ladder on the right, the node voltages at f and b are equal. Thus, there is no current flowing through the resistor R_1

$$i_{R1} = \frac{u_f - u_b}{R_1} = \frac{0\text{V}}{R_1} = 0\text{A}$$

- ii. Select the correct statement regarding the relative values of i_{R1} and i_{R2} .

$i_{R1} > i_{R2}$

$i_{R1} = i_{R2}$

$i_{R1} < i_{R2}$

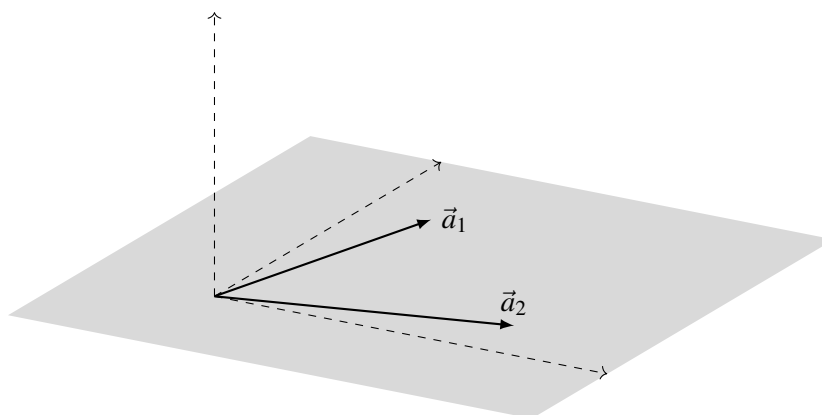
Solution:

Using a similar justification as i_{R1} in part (c).i., the node voltages at g and c are equal, thus $i_{R2} = 0\text{A}$.

Consequently, $i_{R1} = i_{R2} = 0\text{A}$.

4. Least Squares in Pictures (9 points)

- (a) (1 point) Consider a 3×2 matrix \mathbf{A} with linearly independent columns $\vec{a}_1, \vec{a}_2 \in \mathbb{R}^3$. Below we plot \vec{a}_1 and \vec{a}_2 as well as the plane that they both lie in (the dashed lines are the coordinate axes).



Select all of the following options which describe the plane that contains \vec{a}_1 and \vec{a}_2 (the shaded region):

- $\text{Col}(\mathbf{A})$, the columnspace of A .
- $\text{Null}(\mathbf{A})$, the nullspace of A .
- $\text{Det}(\mathbf{A})$, the determinant of A .
- $\{\mathbf{A}\vec{x} \mid \vec{x} \in \mathbb{R}^2\}$.

Solution:

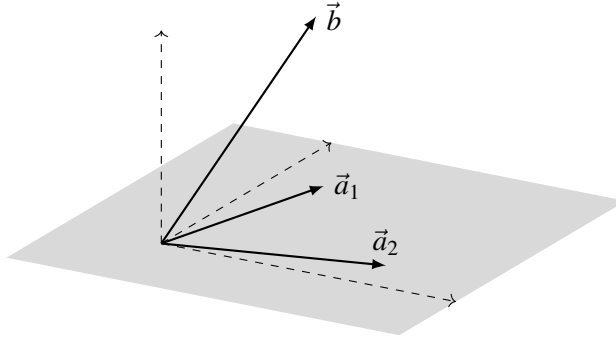
Recall that you can write matrix-vector multiplication as a linear combination of the columns of the matrix: $\mathbf{A}\vec{x} = \sum_{i=1}^n x_i \vec{a}_i$. Therefore, the possible values of $\mathbf{A}\vec{x}$ for different \vec{x} (the last option) are exactly the span of the columns of \mathbf{A} , also known as the column space of \mathbf{A} (the first option). In this case, because \vec{a}_1 and \vec{a}_2 are linearly independent, they span a full 2D plane in \mathbb{R}^3 , which is the plane shown in the picture.

In the context of least squares, this plane represents the part of \mathbb{R}^3 that is “reachable” by \mathbf{A} . There is no \hat{x} you can choose to create an $\mathbf{A}\hat{x}$ outside of this plane.

- (b) (2 points) Now let's take our setup from part (a) and add an additional vector $\vec{b} \in \mathbb{R}^3$ to the diagram. For each of the following choices of \vec{b} , select whether the equation $\mathbf{A}\vec{x} = \vec{b}$ has an *exact* solution \vec{x} .

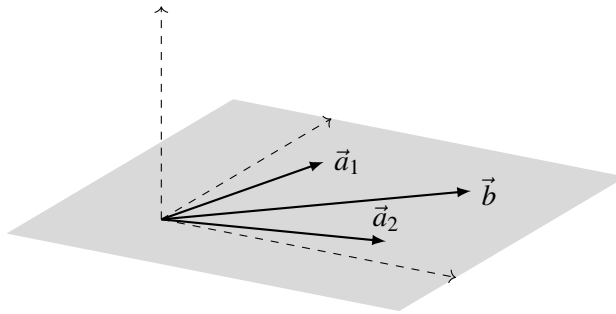
Solution: The system of linear equations $\mathbf{A}\vec{x} = \vec{b}$ has an exact solution if and only if $\vec{b} \in \text{Col}(\mathbf{A})$. Geometrically, that means \vec{b} must lie in the plane of \vec{a}_1 and \vec{a}_2 .

- (i). \vec{b} is *not* in the plane



- Yes, there exists an \vec{x} such that $\mathbf{A}\vec{x} = \vec{b}$.
- No, there does *not* exist an \vec{x} such that $\mathbf{A}\vec{x} = \vec{b}$.

- (ii). \vec{b} is in the plane

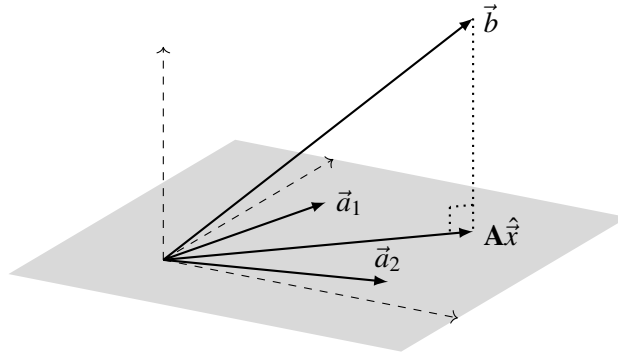


- Yes, there exists an \vec{x} such that $\mathbf{A}\vec{x} = \vec{b}$.
- No, there does *not* exist an \vec{x} such that $\mathbf{A}\vec{x} = \vec{b}$.

- (c) (3 points) Now let's take our setup from the previous parts and add one more vector $\mathbf{A}\hat{\mathbf{x}}$. Determine in each of the following pictures whether the $\hat{\mathbf{x}}$ from the given $\mathbf{A}\hat{\mathbf{x}}$ vector is the least squares solution to $\mathbf{A}\hat{\mathbf{x}} \approx \vec{b}$ or not.

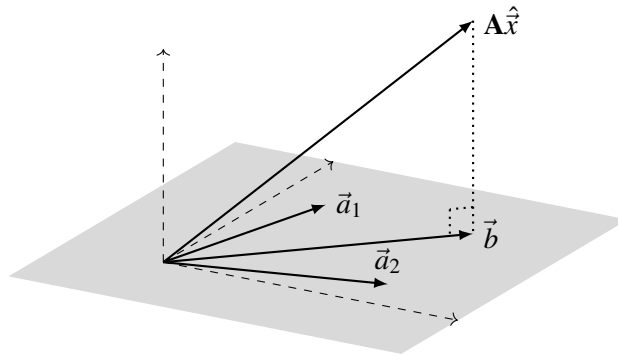
Solution: Geometrically, the least squares solution to $\mathbf{A}\hat{\mathbf{x}} \approx \vec{b}$ is the orthogonal projection of \vec{b} onto the column space of \mathbf{A} because that is the choice of $\mathbf{A}\hat{\mathbf{x}}$ that minimizes the error between $\mathbf{A}\hat{\mathbf{x}}$ and \vec{b} . This corresponds to the picture in part (i).

- (i). \vec{b} is *not* in the plane. $\mathbf{A}\hat{\mathbf{x}}$ is in the plane.



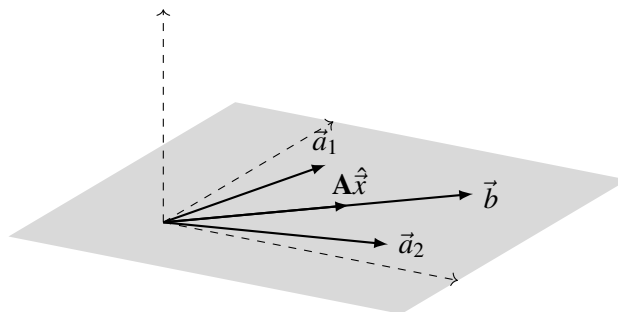
- Yes, $\hat{\mathbf{x}}$ is the least squares solution.
- No, $\hat{\mathbf{x}}$ is *not* the least squares solution.

- (ii). \vec{b} is in the plane. $\mathbf{A}\hat{\mathbf{x}}$ is *not* in the plane.



- Yes, $\hat{\mathbf{x}}$ is the least squares solution.
- No, $\hat{\mathbf{x}}$ is *not* the least squares solution.

- (iii). \vec{b} is in the plane. $\mathbf{A}\hat{\mathbf{x}}$ is in the plane and ends at the tip of its arrow (not all the way to the tip of \vec{b}).



- Yes, $\hat{\vec{x}}$ is the least squares solution.
- No, $\hat{\vec{x}}$ is *not* the least squares solution.

Solution: When \vec{b} happens to be in the plane that is the column space of \mathbf{A} , there is an exact solution where $\mathbf{A}\hat{\vec{x}} = \vec{b}$ that has zero error. So geometrically, $\mathbf{A}\hat{\vec{x}}$ and \vec{b} should be the same vector.

(d) (3 points) Suppose that you have found the least squares solution $\hat{\vec{x}}$ to the equation $\mathbf{A}\vec{x} \approx \vec{b}$. Which of the following statements are always true for any \mathbf{A} and \vec{b} ?

- \vec{b} is orthogonal to $\hat{\vec{x}}$
- \vec{b} is orthogonal to $\mathbf{A}\hat{\vec{x}}$
- $\vec{b} - \mathbf{A}\hat{\vec{x}}$ is orthogonal to \vec{a}_1
- $\vec{b} - \mathbf{A}\hat{\vec{x}}$ is orthogonal to every vector $\vec{y} \in \text{Col}(\mathbf{A})$
- $\mathbf{A}\hat{\vec{x}}$ is orthogonal to \vec{a}_2
- $\mathbf{A}\hat{\vec{x}}$ is parallel or anti-parallel to \vec{b}

Solution: The orthogonality principle of least squares states that the error of the optimal least squares estimate $\vec{b} - \mathbf{A}\hat{\vec{x}}$ is orthogonal to everything in the space of possible estimates $\mathbf{A}\hat{\vec{x}}$, i.e. the column space of \mathbf{A} .

5. Battleships (12 points)

You are working on a top-secret project to predict the distance you are from enemy ships in the ocean. You want to build a model that makes these predictions based on a variety of factors, such as water temperature and currents. In particular, you want to design a linear function by choosing the best values of c_1, c_2 :

$$d = c_1 m_1 + c_2 m_2$$

where m_1, m_2 correspond to measurements of temperature and currents, and d represents your prediction of the enemy ship's distance.

(a) (3 points) You begin by taking the following measurements:

m_1	m_2	d
0	1	0
1	1	1
2	4	-2

You want to use this data to solve for the best parameters of your model. Find \mathbf{A} and \vec{b} such that your problem is in the form $\mathbf{A}\vec{x} = \vec{b}$, where $\vec{x} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$, the parameters of the model to solve for.

$\mathbf{A} =$

$\vec{b} =$

Solution: $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 4 \end{bmatrix}$ $\vec{b} = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$

(b) (5 points) Suppose your colleague took their own measurements, and came up with the following \mathbf{A} matrix and \vec{b} vector.

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\vec{b} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

Explicitly solve for the least-squares solution $\hat{\vec{x}}$.

$\hat{x} =$

Solution:

$$\begin{aligned}
 \hat{x} &= (A^T A)^{-1} A^T \vec{b} \\
 &= \left(\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \\
 &= \left(\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 3 \\ 3 \end{bmatrix} \\
 &= \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 3 \end{bmatrix} \\
 &= \begin{bmatrix} \frac{3}{2} \\ 3 \end{bmatrix}
 \end{aligned}$$

Alternatively, because the columns of A are orthogonal, you can use the result from Discussion 14A Problem 4(b) to just project \vec{b} onto each column of A .

(c) (4 points) Your boss attempts to calculate the least squares solution on their own and arrives at

$$\hat{x} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

Compute the squared error that this model achieves on the data, which is repeated below for your convenience.

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \vec{b} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

Hint: The squared error is defined as the squared norm of the error vector.

squared error =

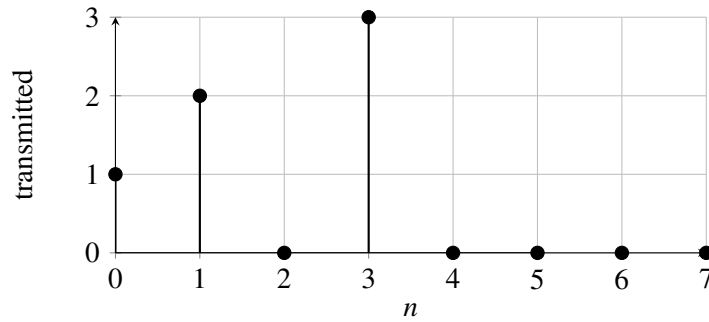
Solution:

$$\begin{aligned}\vec{b} - A\hat{x} &= \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \\ \|\vec{b} - A\hat{x}\|^2 &= 0 + (-1)^2 + 1^2 = 2\end{aligned}$$

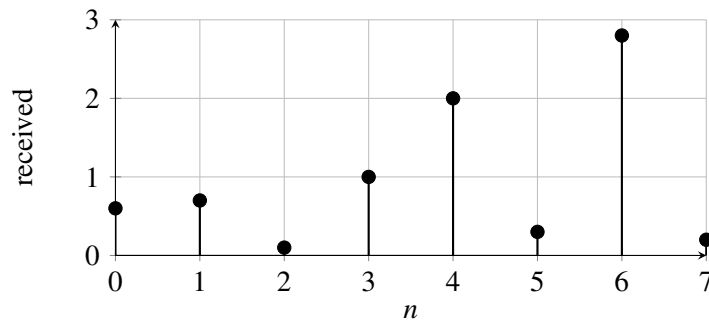
6. 16 Aliens (12 points)

Shuming is stranded on an alien planet and can only receive sound signals from distant signal towers. Can you help him find his way back to the launchpad?

(a) (2 points) One of the towers transmits the following signal:



Shuming receives the following noisy signal:

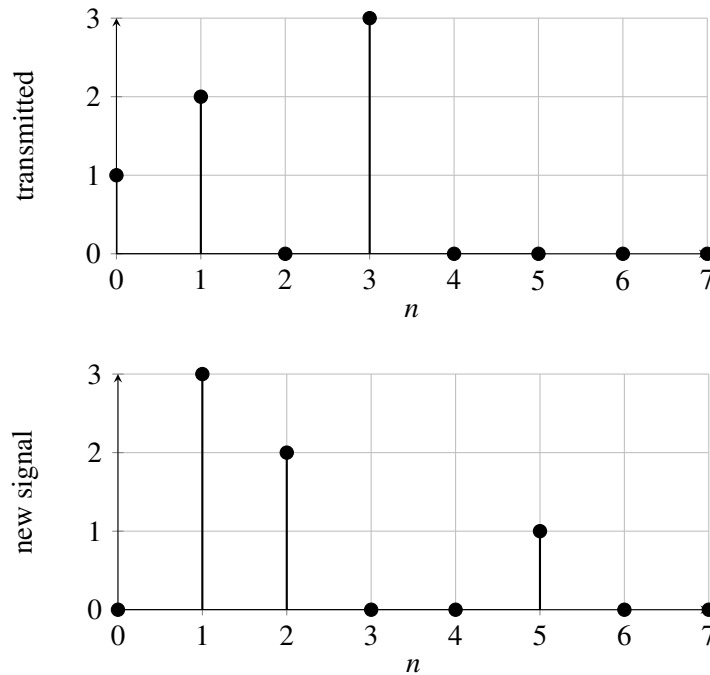


How long did it take for the tower's transmission to reach Shuming? (Assume that the units of n are seconds.)

seconds

Solution: Looking for where the transmitted signal matches with the received signal, it matches best starting at $n = 3$ seconds, so it took 3 seconds to arrive.

(b) (2 points) Shuming received another signal!



Calculate the cross-correlation between the new signal and the same transmitted signal delayed by 1, i.e. $corr_{new\ signal}(transmitted)[1]$.

$$corr_{new\ signal}(transmitted)[1] =$$

Note: $corr_{\vec{x}}(\vec{y})[k] = crosscorr(\vec{x}, \vec{y})[k]$.

Solution: Shift the transmitted signal to the right by 1. Then compute the inner product between the two signals: $crosscorr(new\ signal, transmitted)[1] = 1 \cdot 3 + 2 \cdot 2 = 7$

(c) (1 point) Shuming receives another signal, and detects a match with the tower's signal with a time delay of $n = 20$ seconds. Given that the speed of sound is 350 meters/second, how far away is the signal tower from Shuming?

 meters

Solution: distance = speed * time: $20\ s * (350\ m/s) = 7000\ m$

(d) (3 points) After performing cross-correlation on the signals, Shuming obtains the following absolute distances between him and each of the towers. Using the provided distances d and tower positions (x, y) , write the corresponding **nonlinear** equations for each tower that we could use to solve for Shuming's location. Use r_x and r_y as the variables that we would solve for Shuming's location.

Tower	(x, y)	d
0	(0,0)	1
1	(3,1)	3
2	(-2,1)	2

Solution: We apply the distance equation to each of the three towers:

$$\begin{aligned}\|\vec{r} - \vec{a}_i\|^2 &= (r_x - x_i)^2 + (r_y - y_i)^2 = d_i^2 \\ r_x^2 + r_y^2 &= 1^2 \\ (r_x - 3)^2 + (r_y - 1)^2 &= 3^2 \\ (r_x + 2)^2 + (r_y - 1)^2 &= 2^2\end{aligned}$$

(e) (4 points) Lastly, linearize your nonlinear equations, and write the final system of linear equations in matrix-vector form. You don't need to solve the system for r_x and r_y .

Solution: Starting with the three equations from the previous part:

$$r_x^2 + r_y^2 = 1 \tag{1}$$

$$r_x^2 - 6r_x + 9 + r_y^2 - 2r_y + 1 = 9 \tag{2}$$

$$r_x^2 + 4r_x + 4 + r_y^2 - 2r_y + 1 = 4 \tag{3}$$

We linearize by subtracting equations from each other and getting a system of two linear equations. Here, we show subtracting the second and third tower's equations from the first, but you can choose other subtractions.

$$\begin{aligned}(2) - (1) : -6r_x + 9 - 2r_y + 1 &= 8 \\ -6r_x - 2r_y &= -2\end{aligned}$$

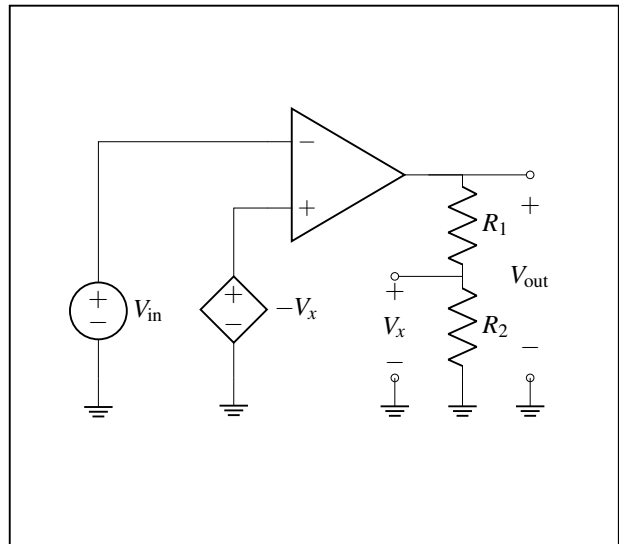
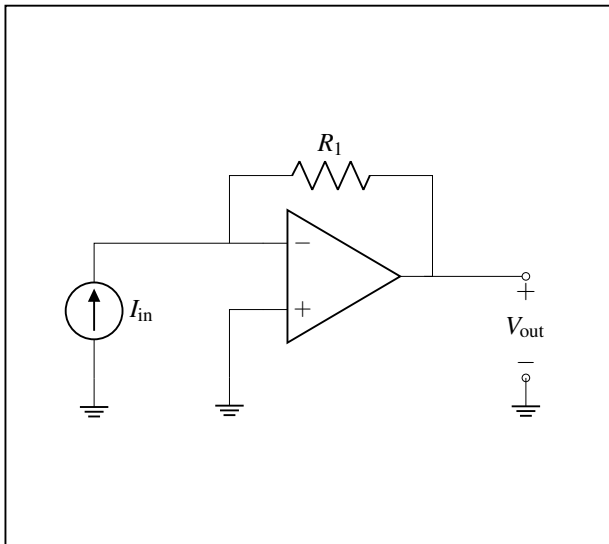
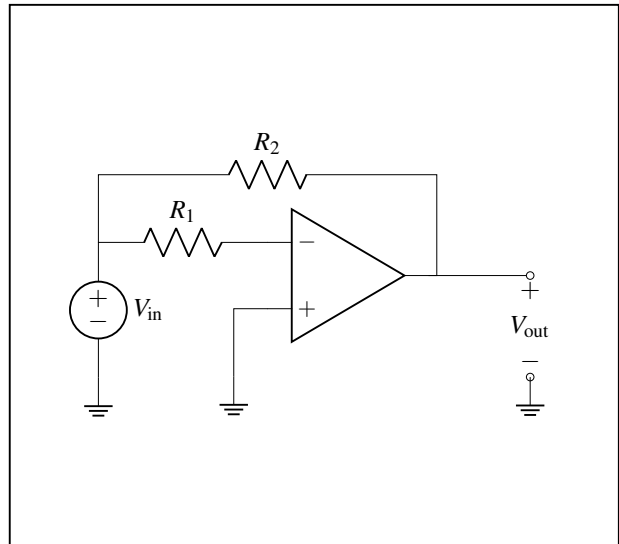
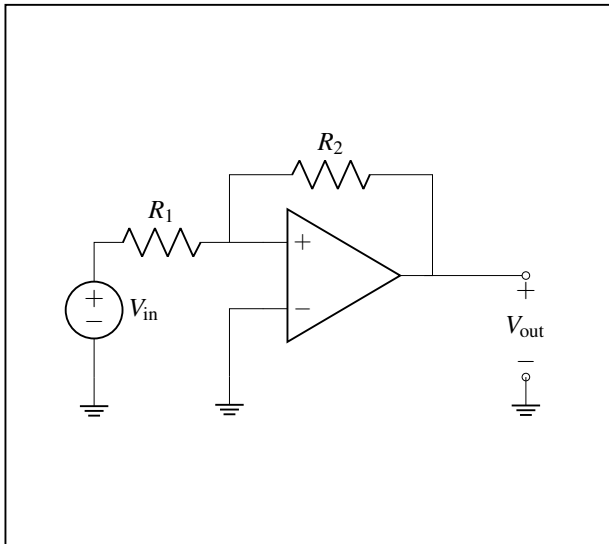
$$\begin{aligned}(3) - (1) : 4r_x + 4 - 2r_y + 1 &= 3 \\ 4r_x - 2r_y &= -2\end{aligned}$$

We can then put this into matrix-vector form:

$$\begin{aligned}A\vec{r} &= \vec{b} \\ \begin{bmatrix} -6 & -2 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} r_x \\ r_y \end{bmatrix} &= \begin{bmatrix} -2 \\ -2 \end{bmatrix}\end{aligned}$$

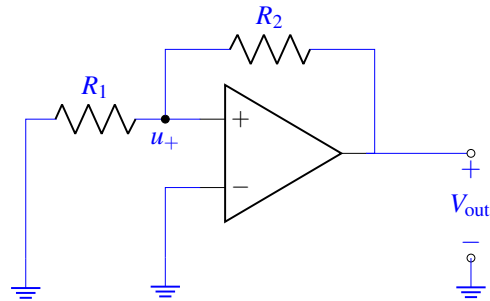
7. Negative Feedback (8 points)

Which of the following op-amp circuits are in negative feedback?

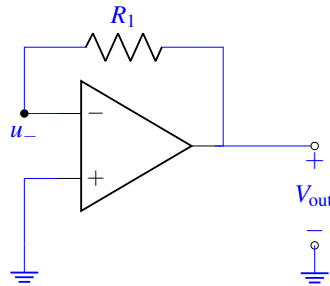


Solution:

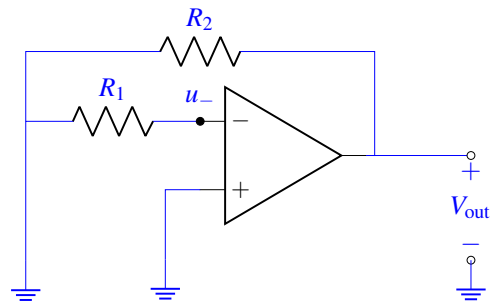
When testing for negative feedback, we need to zero out all independent sources. Let's analyze the circuits where we replace voltage sources with short circuits and current sources with open circuits.



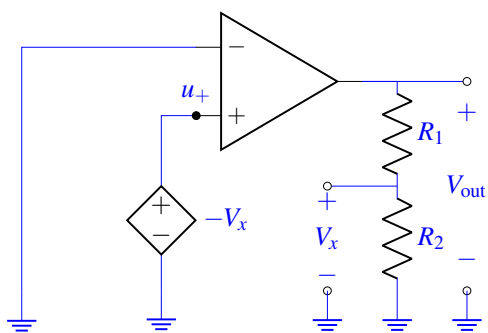
We see that u_+ is the middle node of a voltage divider. If V_{out} increases, so will u_+ which in turn increases the output of the op-amp. Since an increase in V_{out} causes V_{out} to increase even more, this is an example of positive feedback, not negative feedback.



No current flows through R_1 which means that there is no voltage drop across of it and thus $u_- = V_{out}$. An increase to V_{out} will increase u_- which in turn decreases the output of the op-amp. Since an increase in V_{out} causes V_{out} to decrease, this is an example of negative feedback.



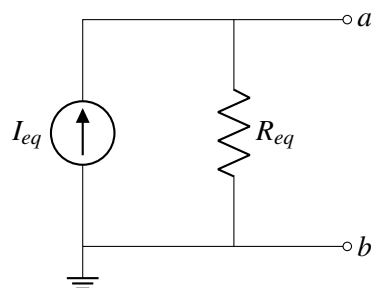
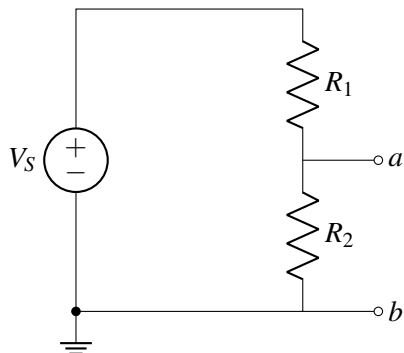
Again, no current flows through R_1 which means that $u_- = 0\text{V}$. Thus any changes to V_{out} does not affect the inputs of the op-amp. This op-amp is not in feedback!



Remember that we must leave the dependent source on! Here we notice that V_x is the middle node of a voltage divider. If V_{out} increases, so will V_x . The increase in V_x affects the dependent source which lowers the voltage at u_+ . Lowering the positive input of the op-amp will decrease the op-amp output. Since an increase in V_{out} causes V_{out} to decrease, this is an example of negative feedback.

8. What's the Equivalent? (12 points)

- (a) (4 points) Solve for the values of I_{eq}, R_{eq} such that the $I - V$ characteristics between terminals a and b are the same in both circuits. Express your answers in terms of R_1, R_2, V_S .



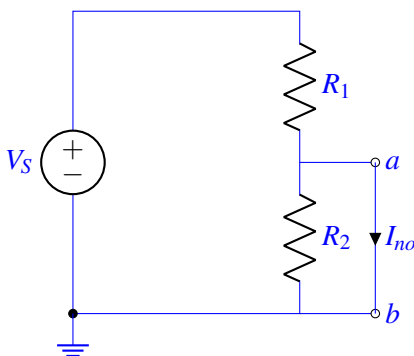
$I_{eq} =$

$R_{eq} =$

Solution:

The circuit on the right is the Norton equivalent circuit! Thus if we find R_{th} and I_{no} for the circuit on the left, we know that $I_{eq} = I_{no}$ and $R_{eq} = R_{th}$.

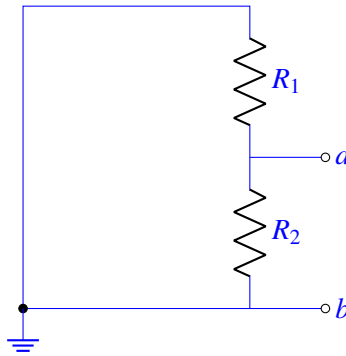
Let's begin by finding I_{no} which is found by shorting nodes a to b and calculating the current flowing through the short.



Since a and b are shorted, $V_a = V_b = 0V$ and no current will flow through R_2 . Thus

$$I_{no} = I_{R_1} = \frac{V_S}{R_1}.$$

Now let's find R_{th} by turning off all independent sources and computing the equivalent resistance between the two terminals.



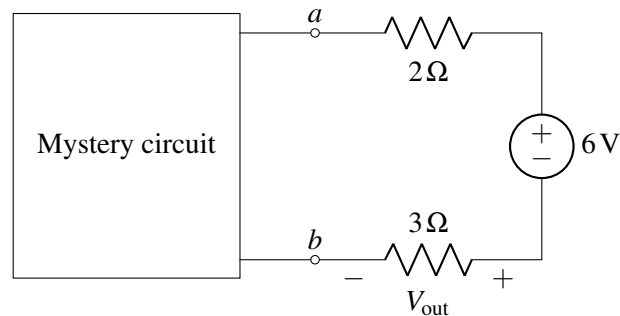
Between nodes a and b , we can see that the two resistors are in parallel. Thus

$$R_{th} = R_1 \parallel R_2.$$

Thus our equivalents are as follows:

$$I_{eq} = I_{no} = \frac{V_S}{R_1} \quad R_{eq} = R_{th} = R_1 \parallel R_2.$$

- (b) (8 points) Consider a mystery circuit. You decide to connect a voltage source and two resistors to its terminals a and b as shown. You are told that the Thévenin equivalent circuit for the mystery circuit between nodes a and b has $V_{th} = 3\text{ V}$ and $R_{th} = 1\ \Omega$.



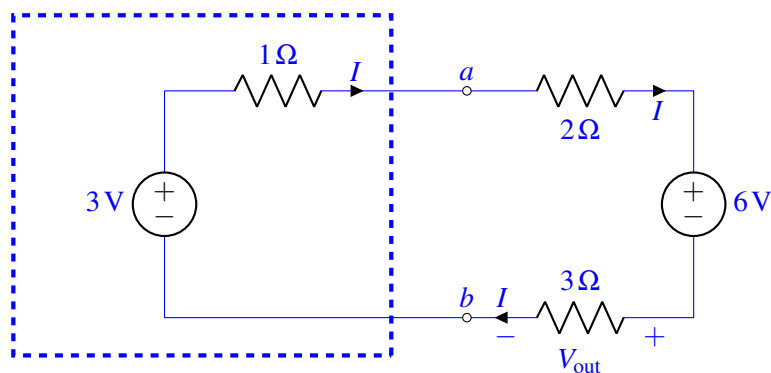
- i. What is the voltage V_{out} across the $3\ \Omega$ resistor?

$$V_{out} = \boxed{} \text{ V}$$

Solution:

Connect the Thévenin equivalent circuit for the mystery circuit:

Mystery Circuit



We notice that all of our circuit elements are in a loop so the current through each element must be the same! Let's label this current I according to passive sign convention with the labeled voltage V_{out} .

Apply KVL in the counter-clockwise direction starting in the bottom left corner using Ohm's law on each resistor:

$$0 = 3\text{ V} - I(1\ \Omega) - I(2\ \Omega) - 6\text{ V} - I(3\ \Omega).$$

Solving for I yields

$$I = \frac{3\text{ V} - 6\text{ V}}{1\ \Omega + 2\ \Omega + 3\ \Omega} = -0.5\text{ A}$$

Now we can use Ohm's law again to find the voltage V_{out} across the $3\ \Omega$ resistor.

$$V_{out} = I \cdot (3\ \Omega) = (-0.5\text{ A})(3\ \Omega) = -1.5\text{ V}$$

ii. What is the power P_{out} dissipated by the 3Ω resistor?

$$P_{\text{out}} = \boxed{} \text{ W}$$

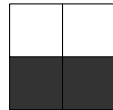
Solution:

We know that power is calculated using $P = IV$. For a resistor, we can use Ohm's law to rewrite

$$P_{\text{out}} = \frac{V_{\text{out}}^2}{R} = \frac{(-1.5\text{ V})^2}{3\Omega} = 0.75\text{ W}.$$

9. Noisy Imaging Revisited (12 points)

(a) (2 points) Oski decides to attend EECS 16A labs! He is given 4 different 2x2 masks for imaging:



Mask 1



Mask 2



Mask 3

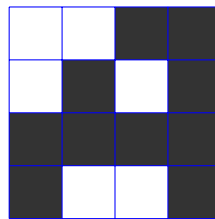


Mask 4

What is the masking matrix that corresponds to the given masks? (Hint: remember that each row of a masking matrix represents a mask.)

Draw the masking matrix by filling in the squares of the grid below (draw an X in the squares that should be black/zero):

Solution: The resultant masking matrix looks like this:



(b) (2 points) Could we use the masking matrix in the previous part to uniquely solve for an image from sensor data? **Remember that black squares represent zeros and white squares represent ones.**

Yes No

Briefly justify your answer.

Solution: No, the masking matrix is not invertible - there is a column (and a row) of zeros. Another way to see this is that the intensity of the fourth pixel is never measured.

- (c) (4 points) Recall that we modeled our noisy imaging system as $\vec{s} = H\vec{i} + \vec{w}$, where \vec{s} is the measurements, H is the masking matrix, \vec{i} is the true scene we want to reconstruct, and \vec{w} is the noise vector. To reconstruct our image, we multiplied both sides of the equation by H^{-1} , yielding $\vec{i}_{est} = \vec{i} + H^{-1}\vec{w}$. The term $H^{-1}\vec{w}$ represents the effect of the noise on \vec{i}_{est} . In order to minimize $H^{-1}\vec{w}$, is it better for the eigenvalues of H to be small or large?

Small Large

Justify your answer. You may assume that the eigenvectors of H span \mathbb{R}^n , where $\vec{w} \in \mathbb{R}^n$.

Solution: We prefer large eigenvalues of H . If the eigenvectors of H span \mathbb{R}^n , we can write $\vec{w} = \sum_{i=1}^n \alpha_i \vec{v}_i$ for some real α_i , where \vec{v}_i are eigenvectors of H . We want to minimize $H^{-1}\vec{w}$, which means minimizing $H^{-1} \sum_{i=1}^n \alpha_i \vec{v}_i = \sum_{i=1}^n \frac{1}{\lambda_i} \alpha_i \vec{v}_i$. To minimize this quantity, we must therefore maximize λ_i .

- (d) (2 points) Another way to counteract the effect of noise is to collect more measurements and take advantage of redundancy using least squares, as you've learned in Module 3. In this case, our new masking matrix H would be tall (more rows than columns) instead of square, so we cannot invert it directly. Let's analyze the effect of noise in this scenario.

First, consider the case when there is no noise in the measurement: $\vec{w} = \vec{0}$ and $\vec{s} = H\vec{i}$. Using least squares, write a mathematical formula for the reconstructed image \vec{i}_{clean} in terms of H and \vec{s} .

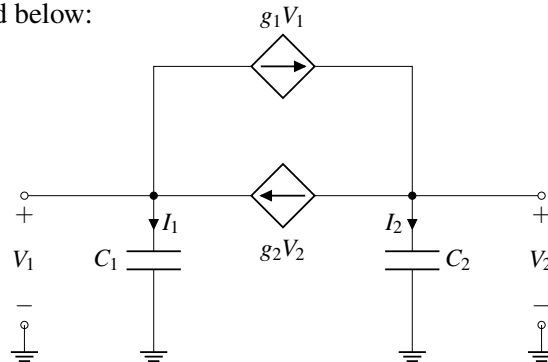
Solution: Our system without noise is $\vec{s} = H\vec{i}$. Using our normal least squares formulation, we get $\vec{i}_{clean} = (H^T H)^{-1} H^T \vec{s}$.

- (e) (2 points) Next, consider the case when there is noise: $\vec{w} \neq \vec{0}$ and $\vec{s} = H\vec{i} + \vec{w}$. To see the effect of noise, use least squares again to write a mathematical formula for the reconstructed image \vec{i}_{noisy} in terms of H , \vec{w} , and \vec{i}_{clean} from the previous part.

Solution: Again using the least squares formula, we get $\vec{i}_{noisy} = (H^T H)^{-1} H^T (\vec{s} - \vec{w})$. This reduces to $\vec{i}_{clean} - (H^T H)^{-1} H^T \vec{w}$.

10. Charge Pumps (15 points)

Consider the circuit described below:



Two capacitors C_1 and C_2 are connected by two dependent current sources which control the flow of current between the capacitors. The charges on the capacitors can be modeled as discrete time signals $Q_1[t]$ and $Q_2[t]$ that are sampled every Δt seconds.

(a) (4 points) We are given that $Q_1[t], Q_2[t]$ change in time according to the following equations:

$$Q_1[t+1] = \left(1 - \frac{g_1 \Delta t}{C_1}\right) Q_1[t] + \left(\frac{g_2 \Delta t}{C_2}\right) Q_2[t]$$

$$Q_2[t+1] = \left(\frac{g_1 \Delta t}{C_1}\right) Q_1[t] + \left(1 - \frac{g_2 \Delta t}{C_2}\right) Q_2[t]$$

Let us define the vector

$$\vec{q}[t] = \begin{bmatrix} Q_1[t] \\ Q_2[t] \end{bmatrix}$$

that represents the charges on both capacitors at time step t .

- i. Assuming $C_1 = 10 \mu\text{F}$, $C_2 = 20 \mu\text{F}$, $g_1 = 5 \text{ A} \cdot \text{V}^{-1}$, $g_2 = 4 \text{ A} \cdot \text{V}^{-1}$, $\Delta t = 1 \mu\text{s}$, find the state transition matrix S where $\vec{q}[t+1] = S\vec{q}[t]$. Note: $1 \text{ F} = 1 \text{ A} \cdot \text{s} \cdot \text{V}^{-1}$.

Solution:

Let's begin by computing

$$\frac{g_1 \Delta t}{C_1} = \frac{5 \text{ A} \cdot \text{V}^{-1} \cdot 1 \mu\text{s}}{10 \mu\text{F}} = \frac{1}{2}$$

$$\frac{g_2 \Delta t}{C_2} = \frac{4 \text{ A} \cdot \text{V}^{-1} \cdot 1 \mu\text{s}}{20 \mu\text{F}} = \frac{1}{5}$$

which allows us to rewrite our given state equations as

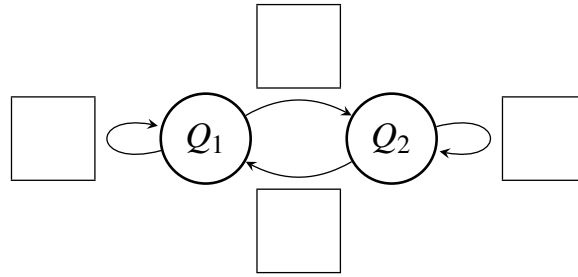
$$Q_1[t+1] = \frac{1}{2} Q_1[t] + \frac{1}{5} Q_2[t]$$

$$Q_2[t+1] = \frac{1}{2} Q_1[t] + \frac{4}{5} Q_2[t].$$

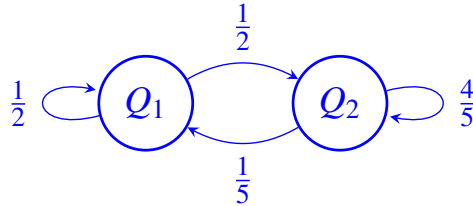
Now converting this system of linear equations into matrix form, we have

$$\begin{bmatrix} Q_1[t+1] \\ Q_2[t+1] \end{bmatrix} = \begin{bmatrix} 1/2 & 1/5 \\ 1/2 & 4/5 \end{bmatrix} \begin{bmatrix} Q_1[t] \\ Q_2[t] \end{bmatrix} \implies S = \begin{bmatrix} 1/2 & 1/5 \\ 1/2 & 4/5 \end{bmatrix}.$$

ii. Fill in the state transition diagram that corresponds to S .



Solution:



iii. Is this system conservative?

Yes No

Solution:

A transition matrix is conservative if its columns sum to one. For our matrix

$$\frac{1}{2} + \frac{1}{2} = \frac{1}{5} + \frac{4}{5} = 1$$

so our system is conservative.

(b) (7 points) Let's assume we found the following state transition matrix for the charge pump circuit:

$$S = \begin{bmatrix} 1/2 & 1/4 \\ 1/2 & 3/4 \end{bmatrix}$$

Find the eigenvalues and corresponding eigenvectors of S .

$$\lambda_1 = \boxed{} \quad \vec{v}_1 = \boxed{} \quad \lambda_2 = \boxed{} \quad \vec{v}_2 = \boxed{}$$

Solution:

We begin by solving for the eigenvalues of our system by solving the characteristic equation of S .

$$\begin{aligned} \det(S - \lambda I) &= 0 \\ \det \begin{bmatrix} \frac{1}{2} - \lambda & \frac{1}{4} \\ \frac{1}{2} & \frac{3}{4} - \lambda \end{bmatrix} &= 0 \\ \left(\frac{1}{2} - \lambda\right) \left(\frac{3}{4} - \lambda\right) - \left(\frac{1}{4}\right) \left(\frac{1}{2}\right) &= 0 \\ \frac{3}{8} - \frac{5}{4}\lambda + \lambda^2 - \frac{1}{8} &= 0 \\ \lambda^2 - \frac{5}{4}\lambda + \frac{1}{4} &= 0 \\ (\lambda - 1) \left(\lambda - \frac{1}{4}\right) &= 0 \end{aligned}$$

which gives us

$$\lambda_1 = 1, \quad \lambda_2 = \frac{1}{4}.$$

Let's now solve for eigenvector \vec{v}_1 that corresponds to $\lambda_1 = 1$. We know that

$$S\vec{v}_1 = \lambda_1\vec{v}_1$$

which we can rearrange as

$$\begin{aligned} S\vec{v}_1 - \lambda_1 I\vec{v}_1 &= 0 \\ (S - \lambda_1 I)\vec{v}_1 &= 0 \\ \begin{bmatrix} -\frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & -\frac{1}{4} \end{bmatrix} \vec{v}_1 &= 0. \end{aligned}$$

We can solve this matrix equation (either using gaussian elimination or by inspection) and find that

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

We can repeat this process to solve for eigenvector \vec{v}_2 that corresponds to $\lambda_2 = 1/4$.

$$\begin{aligned} S\vec{v}_2 &= \lambda_2\vec{v}_2 \\ S\vec{v}_2 - \lambda_2 I\vec{v}_2 &= 0 \\ (S - \lambda_2 I)\vec{v}_2 &= 0 \\ \begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \vec{v}_2 &= 0. \end{aligned}$$

which has solution

$$\vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

(c) (4 points) For the next parts, assume that the charge pump state transition matrix is

$$S = \begin{bmatrix} 2/5 & 1/5 \\ 3/5 & 4/5 \end{bmatrix}$$

which has eigenvalue, eigenvector pairs:

$$\lambda_1 = 1, v_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \quad \lambda_2 = \frac{1}{5}, v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

i. If we start in the state

$$\vec{q}[0] = \alpha\vec{v}_1 + \beta\vec{v}_2$$

where $\alpha \neq 0, \beta \neq 0$, does $\lim_{t \rightarrow \infty} \vec{q}[t]$ converge? Explain.

Yes No

Solution:

We know that $\vec{q}[t+1] = S\vec{q}[t]$, so we can write

$$\vec{q}[t] = S^t \vec{q}[0] = S^t (\alpha\vec{v}_1 + \beta\vec{v}_2) = \alpha S^t \vec{v}_1 + \beta S^t \vec{v}_2.$$

Because \vec{v}_1, \vec{v}_2 are eigenvectors, we can rewrite

$$\vec{q}[t] = \alpha S^t \vec{v}_1 + \beta S^t \vec{v}_2 = \alpha \lambda_1^t \vec{v}_1 + \beta \lambda_2^t \vec{v}_2.$$

Now let's consider

$$\lim_{t \rightarrow \infty} \vec{q}[t] = \lim_{t \rightarrow \infty} (\alpha \lambda_1^t \vec{v}_1 + \beta \lambda_2^t \vec{v}_2).$$

Since our values of $\lambda_1, \lambda_2 \leq 1$,

$$\lim_{t \rightarrow \infty} \vec{q}[t] = \lim_{t \rightarrow \infty} \left(\alpha 1^t \vec{v}_1 + \beta \left(\frac{1}{4}\right)^t \vec{v}_2 \right) = \alpha \vec{v}_1$$

so the limit converges!

- ii. We are given that the starting charges in the capacitors are $Q_1[0] = 10 \mu\text{C}$, $Q_2[0] = 6 \mu\text{C}$. What is $\lim_{t \rightarrow \infty} \vec{q}[t]$?

Solution:

Notice that the columns of S sum to zero so our matrix is conservative! This means that the total charge

$$Q_{tot} = Q_1[0] + Q_2[0] = 16 \mu\text{C}$$

remains constant. From the previous part, we know that the steady state solution will be of the form

$$\alpha \vec{v}_1 = \alpha \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} \alpha \\ 3\alpha \end{bmatrix}.$$

The total charge is

$$Q_{tot} = 16 \mu\text{C} = \alpha + 3\alpha \implies \alpha = 4 \mu\text{C}$$

Thus we will have

$$\lim_{t \rightarrow \infty} \vec{q}[t] = \alpha \vec{v}_1 = \begin{bmatrix} 4 \mu\text{C} \\ 12 \mu\text{C} \end{bmatrix}.$$

11. Fun with EECS16(AB) (12 points)

- (a) (5 points) Suppose that A, B , are in $\mathbb{R}^{n \times n}$. Prove that if AB is invertible, then BA is invertible.

Hint: For matrices A, B , in $\mathbb{R}^{n \times n}$, the following fact holds:

$$\det(A)\det(B) = \det(AB)$$

Solution: The determinant of an invertible matrix is not zero, and the determinant of a non-invertible matrix is zero.

So if AB is invertible, $\det(AB) \neq 0$, which implies that both $\det(A) \neq 0$ and $\det(B) \neq 0$.

Then, if we consider matrix BA , $\det(BA) = \det(B)\det(A) \neq 0$, so BA must also be invertible. \square

- (b) (7 points) Prove the following theorem: If $A \in \mathbb{R}^{n \times m}$ and $B \in \mathbb{R}^{m \times n}$ for $n > m$, then AB is not invertible. In other words, if A is a tall matrix and B is a wide matrix, then their product AB is not invertible.

Hint 1: What must be true about the nullspace of a matrix if it is not invertible?

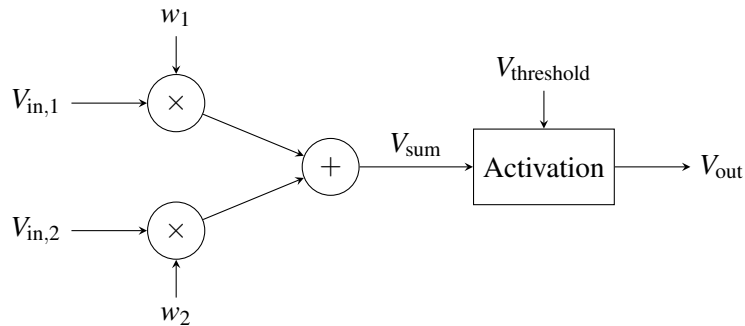
Hint 2: What must be true about the nullspace of a wide matrix?

Solution: First, we argue that a wide matrix like B must have a nontrivial nullspace. Consider the system of equations $B\vec{x} = \vec{0}$. With more columns than rows, there could never be a pivot in every column. Therefore, the system would always have infinite solutions, including nonzero ones, meaning B has a nontrivial nullspace.

Now we want to show that AB is not invertible by showing that it also has a nontrivial nullspace. Consider the equation $(AB)\vec{x} = A(B\vec{x}) = \vec{0}$. If B has a nontrivial nullspace, there exists some nonzero \vec{x} such that $B\vec{x} = \vec{0}$. If we plug in that \vec{x} , we would have $AB\vec{x} = A \cdot \vec{0} = \vec{0}$ for some nonzero \vec{x} , showing that AB has a nontrivial nullspace. \square

12. Artificial Neuron (13 points)

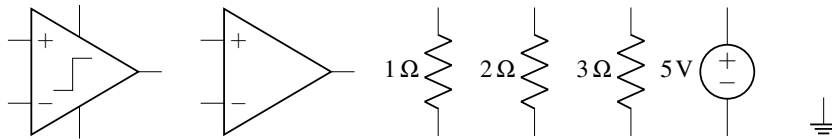
EECS16A is trying to build an AI that can teach the course without any professors or TAs. You are tasked with designing the artificial neuron circuit below that will be used as a core component of the EECS16A AI neural network:



- (a) (8 points) Let's design the first part of the neuron which is a weighted summer $V_{\text{sum}} = w_1 V_{\text{in},1} + w_2 V_{\text{in},2}$. For our neuron, we are given **negative weights** $w_1 = -1.5$, $w_2 = -1$. Design a circuit that implements

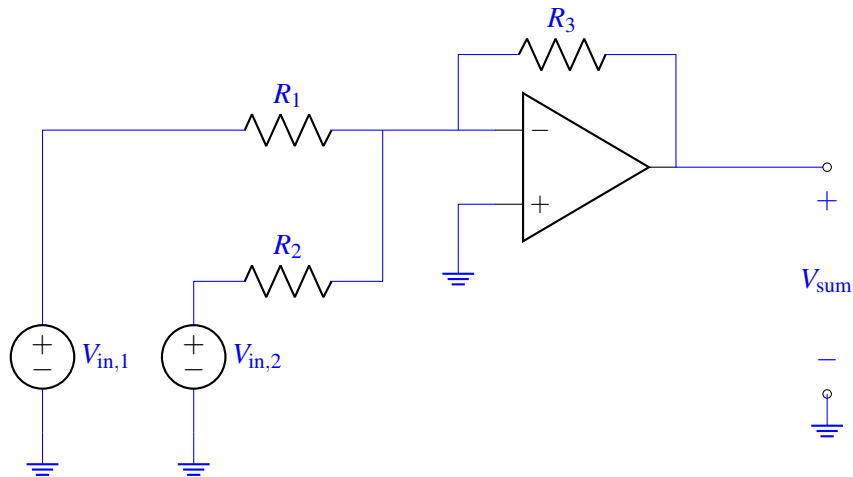
$$V_{\text{sum}} = -1.5V_{\text{in},1} - V_{\text{in},2}$$

using the components below. **You may use multiple of each component. You may not need to use every component.**



Solution:

There are several ways to implement an inverting summer. Here we will go over the simplest solution which we have seen in HW12 P4.



The output of this circuit is given by

$$V_{\text{sum}} = -\frac{R_3}{R_1}V_{\text{in},1} - \frac{R_3}{R_2}V_{\text{in},2}.$$

A quick way to verify this is by using superposition. Notice that if we only turn on one source and short the other, the above circuit just becomes an inverting amplifier! Thus the inverting summer above superposes the outputs of two inverting amplifiers.

We need to select values of R_1, R_2, R_3 such that

$$w_1 = -1.5 = -\frac{R_3}{R_1} \quad w_2 = -1 = -\frac{R_3}{R_2}.$$

Using the list of possible resistor values, we see that we can choose

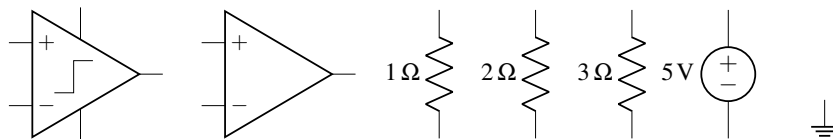
$$R_1 = 2\Omega, R_2 = 3\Omega, R_3 = 3\Omega$$

to satisfy the equations above.

- (b) (4 points) We now need to design the second portion of the circuit. Just like an actual neuron, our artificial neuron should only fire if $V_{\text{sum}} > V_{\text{threshold}}$. When the artificial neuron is inactive, we want $V_{\text{out}} = 0\text{V}$. When the artificial neuron fires, we want $V_{\text{out}} = 5\text{V}$. For our neuron, we are given that $V_{\text{threshold}} = 2.5\text{V}$. Design a circuit that implements

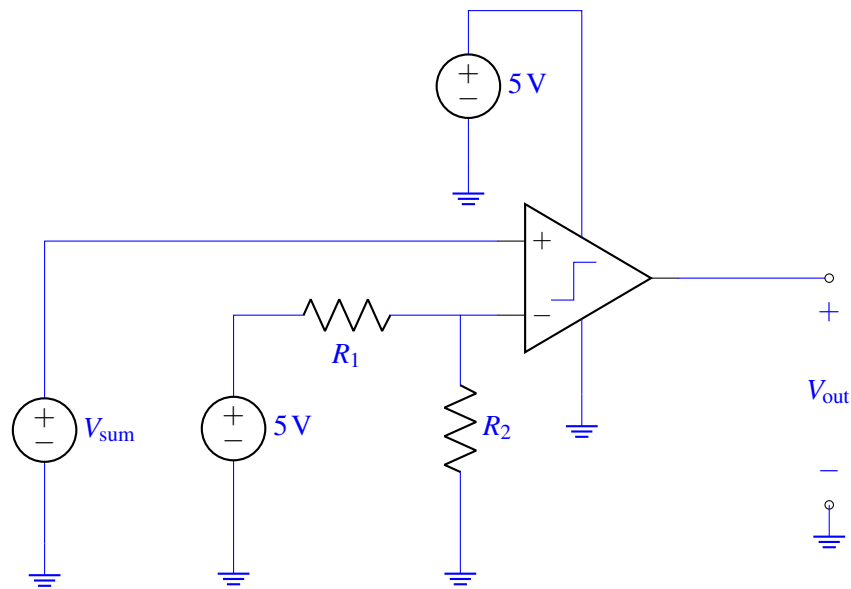
$$V_{\text{out}} = \begin{cases} 0\text{V} & V_{\text{sum}} < 2.5\text{V} \\ 5\text{V} & V_{\text{sum}} > 2.5\text{V} \end{cases}$$

using the components below. **You may use multiple of each component. You may not need to use every component.**



Solution:

Since the output V_{out} is binary (switches between two values) we should use a comparator! The power rails determine the output values so we should connect V_{DD} to 5V and V_{SS} to ground. The output is high when the input $V_{\text{sum}} > V_{\text{threshold}}$ thus V_{sum} should be connected to the positive input and $V_{\text{threshold}}$ to the negative input.



For our circuit, we want to set $V_{\text{threshold}} = 2.5\text{ V}$ but we only have access to a 5 V voltage source. We can use a voltage divider to create a 2.5 V node to connect to the negative input of the comparator. In the circuit above, we need $R_1 = R_2$ which we can set to any of the given resistor values. Ex:

$$R_1 = R_2 = 1\ \Omega.$$

(c) (1 point) When we connect two circuits to one another, we need to be careful about the effects of loading. For the two parts of the artificial neuron circuit we designed, can we directly connect the output of the weighted summer to the input of the thresholding circuit or do we need to add a buffer in between?

- We need a buffer between the two circuits.
- We *do not* need a buffer between the two circuits.

Solution:

The loading effect occurs if the input of the second circuit draws current from the output of the first circuit and changes the output voltage as a result. For the artificial neuron circuit we designed, we notice that the input to the thresholding circuit is connected directly to the comparator which has no input current. This means that even if we connect the two circuits we designed directly, no current will flow between them. As a result, no buffer is needed in between the two circuits.