FIRST Name Eiken LAST Name Wigter

SID (All Digits): 0123456789

- **(5 Points)** On *every* page, print legibly your name and ALL digits of your SID. For every page on which you do not write your name and SID, you forfeit a point, up to the maximum 5 points.
- **(10 Points) (Pledge of Academic Integrity)** Hand-copy, sign, and date the single-line text (which begins with *I have read, ...*) of the Pledge of Academic Integrity on page 3 of this document. Your solutions will *not* be evaluated without this.
- **Urgent Contact with the Teaching Staff:** In case of an urgent matter, raise your hand if in-person, or send an email to `eecs16a@berkeley.edu` if online.
- **This document consists of pages numbered 1 through 14.** Verify that your copy of the exam is free of anomalies, and contains all of the specified number of pages. If you find a defect in your copy, contact the teaching staff immediately.
- This exam is designed to be completed within 70 minutes. However, you may use up to 80 minutes total—*in one sitting*—to tackle the exam.

  The exam starts at 6:40 pm California time. Your allotted window begins with respect to this start time. Students who have official accommodations of $1.5\times$ and $2\times$ time windows have 120 and 160 minutes, respectively.
- **This exam is open book.** So long as you do not violate UC Berkeley, state, or US Federal regulations, you may—during the exam—access or use (a) any reference in print or electronic form; and (b) any computing, communication, or other electronic device, to advance or verify your solutions.

  Citing the output of any such reference does *not* discharge from you the responsibility to provide sufficient explanation for your work.

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- **Collaboration and consultation <u>are</u> permitted**, but
  - *only* with your peers who are currently enrolled in this course;
  - *only* on the public forum of the official Ed Stem page for this course; and
  - *only* to the extent where your peers are unstuck (i.e., not to the level where the solution becomes trivial or otherwise obvious to those who read your hints). Please use good judgment in dispensing hints to each other.

  No private communication is allowed with anyone.

  Communication in any other form—oral, written, or electronic; public or private; direct or indirect—with any human being outside the scope permitted above is *not* permitted.

  Except for staff-approved in-person settings, you may *not* work on this exam in the physical proximity of any student currently enrolled in this course.

- Please write neatly and legibly, because *if we can't read it, we can't evaluate it.*

- For each problem, limit your work to the space provided specifically for that problem. *No other work will be considered. For example, we will not evaluate scratch work. No exceptions.*

- Unless explicitly waived by the specific wording of a problem, you must explain your responses (and reasoning) succinctly, but clearly and convincingly.

- In some parts of a problem, we may ask you to establish a certain result—for example, "show this" or "prove that." Even if you're unable to establish the result that we ask of you, you may still take that result for granted—and use it in any subsequent part of the problem.

- If we ask you to provide a "reasonably simple expression" for something, by default we expect your expression to be in closed form—one *not* involving a sum $\sum$ or an integral $\int$—*unless* we explicitly tell you otherwise.

- Noncompliance with these or other instructions from the teaching staff—*including, for example, commencing work prematurely, or continuing it beyond the allocated time window*—is a serious violation of the Code of Student Conduct.

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Pledge of Academic Integrity

By my honor, I affirm that

(1) this document—which I have produced for the evaluation of my performance—reflects my original, bona fide work, and that I have neither provided to, nor received from, anyone excessive or unreasonable assistance that produces unfair advantage for me or for any of my peers;

(2) as a member of the UC Berkeley community, I have acted with honesty, integrity, respect for others, and professional responsibility—and in a manner consistent with the letter and intent of the campus Code of Student Conduct;

(3) I have not violated—nor aided or abetted anyone else to violate—the instructions for this exam given by the course staff, including, but not limited to, those on the cover page of this document; and

(4) More generally, I have not committed any act that violates—nor aided or abetted anyone else to violate—UC Berkeley, state, or Federal regulations, during this exam.

(10 Points) In the space below, hand-write the following sentence, verbatim. Then write your name in legible letters, sign, include your full SID, and date before submitting your work:

I have read, I understand, and I commit to adhere to the letter and spirit of the pledge above.

I have read, I understand, and I commit to adhere to the letter and spirit of the pledge above.

Full Name: Signature:

Date: 24 Apr 2024 Student ID: 0123456789

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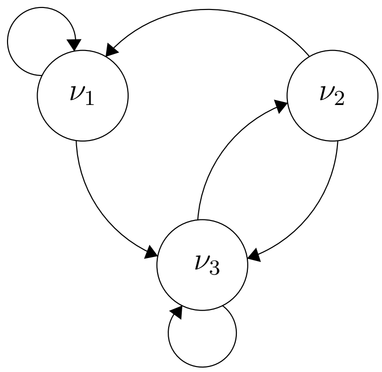
MT3.1 (40 Points) Consider the directed graph and its corresponding adjacency matrix, shown in the diagram below:



(a) A directed graph.

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \left[\underline{a}_1 \ \underline{a}_2 \ \underline{a}_3\right]$$

(b) The Adjacency Matrix of the Graph

Figure 1: A directed graph and its adjacency matrix.

Determine a basis for each of the four subspaces of the adjacency matrix $\mathbf{A}$. You must provide a succinct, yet clear and convincing explanation for each part.

(a) (10 Points) $\mathcal{N}(\mathbf{A})$, the null space of $\mathbf{A}$. The first two columns are linearly dependent: $\underline{a}_1 = \underline{a}_2$. So $A\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \underline{a}_1 - \underline{a}_2 = \underline{0} \Rightarrow$

$$\boxed{\mathcal{N}(A) = \text{span}\left(\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}\right).}$$

(b) (10 Points) $\mathcal{C}(\mathbf{A})$, the column space of $\mathbf{A}$. Either $\text{span}(\underline{a}_1, \underline{a}_3)$ or $\text{span}(\underline{a}_2, \underline{a}_3)$ works.

$$\boxed{C(A) = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right)}$$

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MT3.1

(c) (10 Points) $\mathcal{C}(\mathbf{A}^\top)$, the row space of $\mathbf{A}$.

Viewing A in terms of its rows, we have

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} \underline{\alpha}_1^T \\ \underline{\alpha}_2^T \\ \underline{\alpha}_3^T \end{bmatrix} \Longrightarrow A^T = \begin{bmatrix} \underline{\alpha}_1 & \underline{\alpha}_2 & \underline{\alpha}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

A has two linearly independent rows—equivalently, A^T has two linearly independent columns.

Any of the following descriptions works:

$$C(A^T) = \text{span}(\underline{\alpha}_1, \underline{\alpha}_2) = \text{span}(\underline{\alpha}_1, \underline{\alpha}_3) = \text{span}(\underline{\alpha}_2, \underline{\alpha}_3)$$

$$\boxed{C(A^T) = \text{span}\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \text{span}\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right) = \text{span}\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right)}$$

(d) (10 Points) $\mathcal{N}(\mathbf{A}^\top)$, the left null space of $\mathbf{A}$.

Viewing A in terms of its rows, we have

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} \underline{\alpha}_1^T \\ \underline{\alpha}_2^T \\ \underline{\alpha}_3^T \end{bmatrix} \Longrightarrow A^T = \begin{bmatrix} \underline{\alpha}_1 & \underline{\alpha}_2 & \underline{\alpha}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Note that $\underline{\alpha}_1^T + \underline{\alpha}_2^T = \underline{\alpha}_3^T \Longrightarrow \underline{\alpha}_1^T + \underline{\alpha}_2^T - \underline{\alpha}_3^T = \underline{0}^T$ (Equivalently $\underline{\alpha}_1 + \underline{\alpha}_2 - \underline{\alpha}_3 = \underline{0}$)

Accordingly $\begin{bmatrix} 1 & 1 & -1 \end{bmatrix} A = \underline{\alpha}_1^T + \underline{\alpha}_2^T - \underline{\alpha}_3^T = \underline{0}^T \Longrightarrow$

$$\boxed{C(A^T) = Row(A) = \text{span}\left(\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}\right)}$$

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MT3.2 (50 Points) Consider a real-valued 2×2 matrix $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ whose entries a, b, c, and d are known, and whose determinant is positive—that is, $|A| = ad - bc > 0$.

(a) (5 Points) Explain, in as plain an English as possible, why $\mathbf{A}$ is guaranteed to have a $\mathbf{QR}$ decomposition.

The determinant of A is nonzero, so A is invertible. Therefore, A has linearly independent columns. Accordingly, the Gram-Schmidt Algorithm succeeds. So a QR decomposition exists.

(b) (45 Points) Show that the matrix $\mathbf{A}$ has the $\mathbf{QR}$ decomposition of the form

$$\mathbf{A} = \underbrace{\tau \begin{bmatrix} a & \varphi \\ c & a \end{bmatrix}}_{\mathbf{Q}} \underbrace{\begin{bmatrix} \mu & \lambda \\ 0 & \gamma \end{bmatrix}}_{\mathbf{R}},$$

where $\mathbf{Q}$ is an orthogonal matrix, and τ, φ, μ, λ, and γ are real parameters that you must determine in terms of the known parameters a, b, c, and d.

Please note that the parameters τ, μ, and γ must be positive quantities.

Hints: Start with τ, then φ, and take it from there. Your expressions should be reasonably simple.

If you're unsure of your expression for φ, you'll get a second chance to determine it when you try to determine the upper-triangular matrix $\mathbf{R}$.

If you're unsure of your expression for τ, you can write your expressions for the entries μ, λ, and γ of the upper-triangular matrix $\mathbf{R}$ in terms of τ. This way, you can receive at least partial, if not full, credit, depending on how close to correct your expressions are.

Write the final form of your matrices $\mathbf{Q}$ and $\mathbf{R}$ in the space below on this page, but use the space on the next page to show your work.

You should be able to do this problem in much less space than allocated.

τ φ

$$\mathbf{Q} = \boxed{\frac{1}{\sqrt{a^2+c^2}}} \begin{bmatrix} a & \boxed{-c} \\ c & a \end{bmatrix} \qquad \mathbf{R} = \frac{1}{\sqrt{a^2+c^2}} \begin{bmatrix} a^2+c^2 & ab+cd \\ 0 & ad-bc \end{bmatrix}$$

The diagonal entries of R are positive: $a^2+c^2>0$, because (1) a & c can't both be zero (otherwise, the first col of A is $\underline{0}$); and (2) we're told that $ad-bc>0$ by the problem statement.

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MT3.2 Use this page to show your work for the **QR** decomposition of **A**.

$$Q = \begin{bmatrix} \tau a & \tau \varphi \\ \tau c & \tau a \end{bmatrix}$$ must be an orthogonal matrix $\Longrightarrow$

q_1, q_2

$$\|q_1\|^2 = \tau^2(a^2+c^2) = 1 \Longrightarrow \tau^2 = \frac{1}{a^2+c^2} \Longrightarrow \boxed{\tau = \frac{1}{\sqrt{a^2+c^2}}}$$

The colums of Q must be orthogonal (and τ doesn't affect orthogonality), so

$$\begin{bmatrix} a \\ c \end{bmatrix} \perp \begin{bmatrix} \varphi \\ a \end{bmatrix} \Longrightarrow \begin{bmatrix} a & c \end{bmatrix}\begin{bmatrix} \varphi \\ a \end{bmatrix} = a\varphi + ac = 0$$

$$\Longrightarrow \varphi + c = 0 \Longrightarrow \boxed{\varphi = -c}$$

Factoid: For any $\underline{v} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \in \mathbb{R}^2$, the vector $\underline{w} = \begin{bmatrix} -\beta \\ \alpha \end{bmatrix}$ is orthogonal to it. $\langle \underline{v}, \underline{w} \rangle = \underline{v}^T \underline{w} = \begin{bmatrix} \alpha & \beta \end{bmatrix}\begin{bmatrix} -\beta \\ \alpha \end{bmatrix} = -\alpha\beta + \alpha\beta = 0$.

We can determine R by noting that $A = QR \Rightarrow R = Q^{-1}A$.
But $Q^{-1} = Q^T \Rightarrow R = Q^T A$.

$$R = \tau \begin{bmatrix} a & c \\ -c & a \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \tau \begin{bmatrix} a^2+c^2 & ab+cd \\ \underbrace{\cancel{-ac+ac}}_{0} & ad-bc \end{bmatrix} \xRightarrow{\tau = \frac{1}{\sqrt{a^2+c^2}}}$$

$$R = \frac{1}{\sqrt{a^2+c^2}} \begin{bmatrix} a^2+c^2 & ab+cd \\ 0 & ad-bc \end{bmatrix}$$

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MT3.3 (50 Points) A real-valued 3×3 matrix is partially described below

$$\mathbf{A} = \begin{bmatrix} 1 & \alpha & \beta \\ \alpha & -1 & \gamma \\ \beta & \gamma & 0 \end{bmatrix},$$

where the entries given by α, β, and γ are unknown.

We have the following additional information about the matrix $\mathbf{A}$:

(1) $\mathbf{1}^\mathsf{T}\mathbf{A} = \mathbf{0}^\mathsf{T}$; and

(2) Of the three eigenvalues λ_1, λ_2, and λ_3, we know that $\lambda_1 = \sqrt{3}$.

(a) (10 Points) Consider the vector

$$\boldsymbol{z} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

Select the strongest correct statement from the choices below:

(I) The vector $\boldsymbol{z}$ *must* be in the null space $\mathcal{N}(\mathbf{A})$.

(II) The vector $\boldsymbol{z}$ *can not* be in the null space $\mathcal{N}(\mathbf{A})$.

(III) The vector $\boldsymbol{z}$ *may or may not* be in the null space $\mathcal{N}(\mathbf{A})$; we need additional information to make the determination.

Provide a succinct, yet clear and convincing explanation for your choice.

If you choose (III), specify what additional information you need to determine whether $\boldsymbol{z} \in \mathcal{N}(\mathbf{A})$.

$$A\underline{z} = \begin{bmatrix} 1 & \alpha & \beta \\ \alpha & -1 & \gamma \\ \beta & \gamma & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \alpha+1 \\ \alpha-1 \\ \beta+\gamma \end{bmatrix}$$

Sum of the first two columns of A.

For $A\underline{z}$ to be $\underline{0}$, we must have $\alpha+1=0$ AND $\alpha-1=0$, which is not possible.

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MT3.3

(b) (10 Points) Without resorting to the computation of the unknown entries α, β, and γ, determine a vector in the null space $\mathcal{N}(\mathbf{A})$.

$$\mathbb{1}^T A = \underline{0}^T \Rightarrow (\mathbb{1}^T A)^T = (\underline{0}^T)^T \Rightarrow A^T \mathbb{1} = \underline{0}.$$

But $A = A^T$ (A is symmetric) $\Rightarrow A\mathbb{1} = \underline{0} \Rightarrow$

$$\mathbb{1} \in \mathcal{N}(A)$$

(c) (10 Points) Compute the numerical values for α, β, and γ to show that

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & -1 & 1 \\ -1 & 1 & 0 \end{bmatrix}.$$

$$\mathbb{1}^T A = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & \alpha & \beta \\ \alpha & -1 & \gamma \\ \beta & \gamma & 0 \end{bmatrix} = \underline{0}^T \Rightarrow \begin{cases} 1+\alpha+\beta = 0 \\ \alpha - 1 + \gamma = 0 \\ \beta + \gamma = 0 \end{cases} \Rightarrow$$

$$\begin{cases} \alpha+\beta = -1 \\ \left.\begin{matrix} \alpha+\gamma = 1 \\ \gamma = -\beta \end{matrix}\right\} \end{cases} \Rightarrow \begin{cases} \alpha+\beta = -1 \\ \alpha - \beta = 1 \end{cases}$$

$$2\alpha = 0 \Rightarrow \left.\begin{matrix} \alpha = 0 \\ \alpha+\beta = -1 \end{matrix}\right\} \Rightarrow \left.\begin{matrix} \beta = -1 \\ \gamma = -\beta \end{matrix}\right\} \Rightarrow \gamma = 1$$

So, $\alpha = 0$, $\beta = -1$, $\gamma = 1$

$$A = \begin{bmatrix} 1 & \alpha & \beta \\ \alpha & -1 & \gamma \\ \beta & \gamma & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & -1 & 1 \\ -1 & 1 & 0 \end{bmatrix}$$

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MT3.3

(d) (10 Points) Determine $\boldsymbol{v}_1$, the eigenvector corresponding to the eigenvalue $\lambda_1 = \sqrt{3}$. **Hint:** It may or may not help you to know that

$$-\left(\sqrt{3}-1\right)\left(\sqrt{3}+2\right) = -\left(\sqrt{3}+1\right).$$

$$\lambda_1 = \sqrt{3} \Rightarrow \lambda_1 I - A = \begin{bmatrix} \sqrt{3}-1 & 0 & 1 \\ 0 & \sqrt{3}+1 & -1 \\ 1 & -1 & \sqrt{3} \end{bmatrix}$$

We must have $\underline{v}_1 \in \mathcal{N}(\lambda_1 I - A)$

Let $\underline{v}_1 = \begin{bmatrix} r \\ s \\ t \end{bmatrix}$. To make $(\lambda_1 I - A)\underline{v}_1 = \underline{0}$, it must be that

$(\sqrt{3}+1)\times 3^{rd}\text{col} + 2^{nd}\text{col}$

$$\begin{cases} (\sqrt{3}-1)r + t = 0 & ① \\ (\sqrt{3}+1)s - t = 0 \Rightarrow t = (\sqrt{3}+1)s & ② \\ r - s + \sqrt{3}t = 0 & ③ \end{cases}$$

Let $s = 1 \Rightarrow t = \sqrt{3}+1$

③ $r - 1 + \sqrt{3}(\sqrt{3}+1) = 0 \Rightarrow r - 1 + 3 + \sqrt{3} = 0 \Rightarrow$

$r = -(2+\sqrt{3}) \Rightarrow$

$$\left(\lambda_1 = \sqrt{3},\ \underline{v}_1 = \begin{bmatrix} -(2+\sqrt{3}) \\ 1 \\ 1+\sqrt{3} \end{bmatrix}\right)$$

Giveaway clue: We can create a 0 middle row entry in $(\lambda_1 I - A)\underline{v}_1$ by scaling the third col of $\lambda_1 I - A$ by $\sqrt{3}+1$, and adding it to the second col.

(e) (10 Points) Determine the remaining two eigenvalues (λ_2 and λ_3) of $\mathbf{A}$.

One of the eigenvalues, say λ_2, is zero because $\mathbb{1}^T A = \underbrace{0}_{\lambda_2} \cdot \mathbb{1}^T = \underline{0}^T$.

So far we have $\boxed{\lambda_1 = \sqrt{3}}$ and $\boxed{\lambda_2 = 0}$. To determine λ_3, we use the trace identity—namely

$$\lambda_1 + \lambda_2 + \lambda_3 = a_{11} + a_{22} + a_{33} = \operatorname{Tr}(A).$$

But $\operatorname{Tr}(A) = 0 \Rightarrow \lambda_1 + \lambda_2 + \lambda_3 = \sqrt{3} + 0 + \lambda_3 = 0 \Rightarrow \boxed{\lambda_3 = -\sqrt{3}}$

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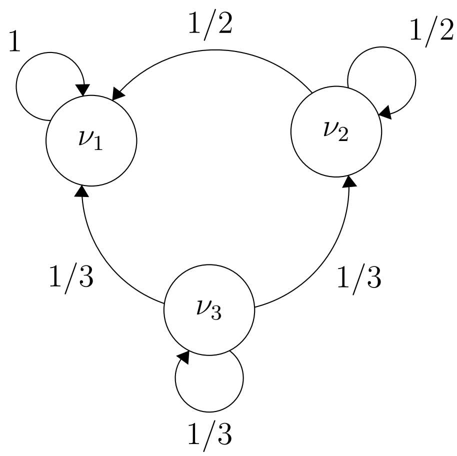
MT3.4 (45 Points) Consider a network of three websites and its corresponding connectivity matrix, shown in the diagram below:



(a) A network of three websites.

$$\mathbf{C} = \begin{bmatrix} 1 & 1/2 & 1/3 \\ 0 & 1/2 & 1/3 \\ 0 & 0 & 1/3 \end{bmatrix}$$

(b) The connectivity matrix of the network.

Figure 2: A network of three websites and its connectivity matrix.

The network is conservative—that is, the total population of Web users is fixed.

We let

$$\boldsymbol{s}(t) = \begin{bmatrix} s_1(t) \\ s_2(t) \\ s_3(t) \end{bmatrix}$$

denote the state vector that captures the distribution of the population of users across the network at time t. The entry $s_i(t)$, for $i = 1, 2, 3$, denotes the fraction of user population at node ν_i at time t.

The vector $\boldsymbol{s}(0)$—called the *initial state*—captures the distribution of users at time $t = 0$. The initial state is a *stochastic vector*, which means that it has the following two properties:

(I) It is a nonnegative vector—that is, $\boldsymbol{s}(0) \succcurlyeq \mathbf{0}$, where $\succcurlyeq$ denotes componentwise inequality; and

(II) The entries of $\boldsymbol{s}(0)$ sum to 1—that is, $\mathbf{1}^\mathsf{T}\boldsymbol{s}(0) = 1$.

The population migration dynamics across the network is captured by the state-evolution equation

$$\boldsymbol{s}(t+1) = \mathbf{C}\boldsymbol{s}(t), \qquad t = 0, 1, 2, \ldots,$$

where the connectivity matrix $\mathbf{C}$ serves as the state-transition matrix.

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MT3.4

(a) (12 Points) We can infer the state at time t from that at time $t+1$ according to the backward state-evolution equation

$$\boldsymbol{s}(t) = \mathbf{B}\boldsymbol{s}(t+1), \qquad t = 0, 1, 2, \ldots.$$

Express how the matrix $\mathbf{B}$ is related to the state-transition matrix $\mathbf{C}$, and compute its entries.

If C is invertible, we can rewrite the state-evolution equation $\underline{s}(t+1) = C\,\underline{s}(t)$ as follows:

$$C^{-1}\underline{s}(t+1) = \underline{s}(t) \implies \underline{s}(t) = \overbrace{C^{-1}}^{B}\underline{s}(t+1) = B\,\underline{s}(t+1)$$

C is upper-triangular with nonzero diagonal entries. We can use back substitution to determine the three columns of B.

$B = [\underline{b}_1\ \underline{b}_2\ \underline{b}_3]$

$C = [\underline{c}_1\ \underline{c}_2\ \underline{c}_3]$

$$\underbrace{\begin{bmatrix} 1 & \frac12 & \frac13 \\ 0 & \frac12 & \frac13 \\ 0 & 0 & \frac13 \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}}_{B} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{I}$$

$\frac13 b_{31} = 0 \Rightarrow \boxed{b_{31} = 0}$; $\frac12 b_{21} + \frac13 \cancelto{0}{b_{31}} = 0 \Rightarrow \boxed{b_{21} = 0}$; $b_{11} + \frac12 \cancelto{0}{b_{21}} + \frac13 \cancelto{0}{b_{31}} = 1 \Rightarrow \boxed{b_{11} = 1}$

You can also argue that the only linear combo of the cols of C that produces the 1st col of I, $\begin{bmatrix}1\\0\\0\end{bmatrix}$, is $\underline{b}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}$. If you saw this, Bravo!

Proceed in a similar fashion with the 2nd and 3rd cols of B.

$\frac13 b_{32} = 0 \Rightarrow \boxed{b_{32} = 0}$; $\frac12 b_{22} + \frac13 \cancelto{0}{b_{32}} = 1 \Rightarrow \boxed{b_{22} = 2}$; $b_{12} + \cancelto{1}{\frac12 b_{22}} + \frac13 \cancelto{0}{b_{32}} = 0$
$\Rightarrow \boxed{b_{12} = -1}$.

$\frac13 b_{33} = 1 \Rightarrow \boxed{b_{33} = 3}$; $\frac12 b_{23} + \cancelto{1}{\frac13 b_{33}} = 0 \Rightarrow \boxed{b_{23} = -2}$; $b_{13} + \cancelto{-1}{\frac12 b_{23}} + \cancelto{1}{\frac13 b_{33}} = 0$
$\Rightarrow \boxed{b_{13} = 0}$

$$\boxed{B = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & -2 \\ 0 & 0 & 3 \end{bmatrix}}$$

Sidenote: It's not a coincidence that the inverse of an upper-triangular matrix is also upper-triangular

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MT3.4

(b) (12 Points) Show that the eigenvector matrix of $\mathbf{C}$ is given by

$$\mathbf{V} = \begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \boldsymbol{v}_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -2 \\ 0 & 0 & 1 \end{bmatrix},$$

and specify the eigenvalue λ_i, $i = 1, 2, 3$, for each eigenvector $\boldsymbol{v}_i$.

$C = \begin{bmatrix} 1 & 1/2 & 1/3 \\ 0 & 1/2 & 1/3 \\ 0 & 0 & 1/3 \end{bmatrix}$ is upper-triangular $\Longrightarrow$

We can read the eigenvalues of C from its diagonal:

$\lambda_1 = 1$, $\lambda_2 = 1/2$, $\lambda_3 = 1/3$.

We already discovered in part (a) that $C \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow$

$\boxed{\left(\lambda_1 = 1,\ \underline{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right)}$ is an eigenpair.

$$C \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \underline{c}_1 - \underline{c}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 1/2 \\ 1/2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ -1/2 \\ 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \Rightarrow \boxed{\left(\lambda_2 = 1/2,\ \underline{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}\right)}$$

$$C \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \underline{c}_1 - 2\underline{c}_2 + \underline{c}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - 2 \begin{bmatrix} 1/2 \\ 1/2 \\ 0 \end{bmatrix} + \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}$$

$$= \begin{bmatrix} 1/3 \\ -2/3 \\ 1/3 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \Rightarrow \boxed{\left(\lambda_3 = \frac{1}{3},\ \underline{v}_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}\right)}$$

We could have asked you to determine the eigenpairs. Again, $\left(\lambda_1=1, \underline{v}_1=\begin{bmatrix}1\\0\\0\end{bmatrix}\right)$ is visible, so we focus on $(\lambda_2=\frac{1}{2}, \underline{v}_2)$ and $(\lambda_3=\frac{1}{3}, \underline{v}_3)$.

$$C=\begin{bmatrix}1 & \frac{1}{2} & \frac{1}{3}\\ 0 & \frac{1}{2} & \frac{1}{3}\\ 0 & 0 & \frac{1}{3}\end{bmatrix}$$

$$\lambda_2 I - C = \frac{1}{2}I - C = \begin{bmatrix}\frac{1}{2}-1 & -\frac{1}{2} & -\frac{1}{3}\\ 0 & \frac{1}{2}-\frac{1}{2} & -\frac{1}{3}\\ 0 & 0 & \frac{1}{2}-\frac{1}{3}\end{bmatrix} = \begin{bmatrix}-\frac{1}{2} & -\frac{1}{2} & -\frac{1}{3}\\ 0 & 0 & -\frac{1}{3}\\ 0 & 0 & \frac{1}{6}\end{bmatrix}$$

$$\uparrow - \uparrow = \underline{0}$$

Clearly the 1st col of $\lambda_2 I - C$ minus the 2nd column produces the zero vector, so $\underline{v}_2=\begin{bmatrix}1\\-1\\0\end{bmatrix} \in \mathcal{N}(\lambda_2 I - C)$.

$\Rightarrow \left(\lambda_2=\frac{1}{2}, \underline{v}_2=\begin{bmatrix}1\\-1\\0\end{bmatrix}\right)$ is an eigenpair.

Proceed in a similar fashion to determine $(\lambda_3, \underline{v}_3)$.

$$\lambda_3 I - C = \begin{bmatrix}\frac{1}{3}-1 & -\frac{1}{2} & -\frac{1}{3}\\ 0 & \frac{1}{3}-\frac{1}{2} & -\frac{1}{3}\\ 0 & 0 & \frac{1}{3}-\frac{1}{3}\end{bmatrix} = \begin{bmatrix}-\frac{2}{3} & -\frac{1}{2} & -\frac{1}{3}\\ 0 & -\frac{1}{6} & -\frac{1}{3}\\ 0 & 0 & 0\end{bmatrix}$$

No matter what $\underline{v}_3$ is, the 3rd entry of $(\lambda_3 I - C)\underline{v}_3$ is zero.

From the yellow-shaded entries we know that in the linear combo of the columns of $\lambda_3 I - C$, the coefficient for the 2nd column must be -2 times that of the 3rd column. Otherwise, we can't produce a zero at the 2nd entry of $(\lambda_3 I - C)\underline{v}_3 = \underline{0}$. So, we try $\underline{v}_3=\begin{bmatrix}\alpha\\-2\\1\end{bmatrix}$. Applying to the 1st row of $\lambda_3 I - C$, we have

$$-\frac{2}{3}\alpha - 2\left(-\frac{1}{2}\right) + \left(-\frac{1}{3}\right) = 0 \Rightarrow -\frac{2}{3}\alpha + 1 - \frac{1}{3} = -\frac{2}{3}\alpha + \frac{2}{3} = 0 \Rightarrow \alpha = 1 \Rightarrow \left(\lambda_3=\frac{1}{3}, \underline{v}_3=\begin{bmatrix}1\\-2\\1\end{bmatrix}\right)$$

is an eigenpair.

FIRST Name Eiken LAST Name Wigter SID (All Digits): 0123456789

MT3.4

(c) (12 Points) Show that the state vector $\boldsymbol{s}(t)$ at any time $t = 0, 1, 2, \ldots$ is stochastic—that is, show that

- $\boldsymbol{s}(t) \succcurlyeq \mathbf{0}$; and
- $\mathbb{1}^\mathsf{T}\boldsymbol{s}(t) = 1$.

$\underline{s}(t+1) = C\,\underline{s}(t) \quad t = 0, 1, 2, \ldots$

Note that C is column stochastic—that is, $C \succcurlyeq 0$ and $\mathbb{1}^T C = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1/2 & 1/3 \\ 0 & 1/2 & 1/3 \\ 0 & 0 & 1/3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} = \mathbb{1}^T$

$\underline{s}(1) = C\,\underline{s}(0) \Rightarrow \mathbb{1}^T \underline{s}(1) = \underbrace{\mathbb{1}^T C}_{=\mathbb{1}^T}\,\underline{s}(0) = \mathbb{1}^T \underline{s}(0) = 1$

$\underline{s}(2) = C\,\underline{s}(1) \Rightarrow \mathbb{1}^T \underline{s}(2) = \mathbb{1}^T C\,\underline{s}(1) = \mathbb{1}^T \underline{s}(1) = 1$

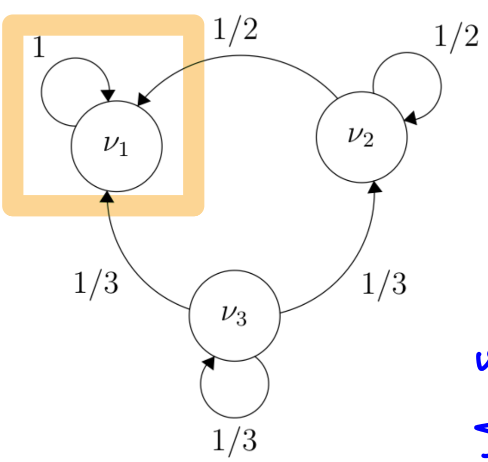
Proceeding in this way, we have $\mathbb{1}^T \underline{s}(t) = 1 \quad \forall t = 0, 1, 2, \ldots$

Furthermore, since both C and $\underline{s}(0)$ are nonnegative, so is $\underline{s}(1) = C\,\underline{s}(0)$ since $\underline{s}(1)$ is simply a nonnegative linear combo of nonnegative column vectors in C. Proceeding in this way, $\underline{s}(2) = C\,\underline{s}(1)$, another linear combo of nonnegative columns. And so on.

(d) (9 Points) With little to no mathematical analysis, determine the limiting state

$$\boldsymbol{s}_\infty = \lim_{t \to \infty} \boldsymbol{s}(t).$$

Provide a succinct, yet clear and convincing explanation.



Method 1:

Note that node ν_1 has no outgoing branches (edges). It's an absorbing node. As $t \to \infty$, every user will end up there. And once there, the user can't leave. So,

$\underline{s}_\infty = \lim_{t \to \infty} \underline{s}(t) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

Method 2 (A bit of Math): The limiting state also happens to be an equilibrium state, which satisfies $\underline{s}(t+1) = C\,\underline{s}(t) = \underline{s}(t)$

So, $\underline{s}(t) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ since this is the eigvec corresponding to eigval 1.