MT1.1 (45 Points) Consider the following vectors in $\mathbb{R}^3$:

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \qquad \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ -1/2 \\ -1/2 \end{bmatrix}, \qquad \text{and} \qquad \boldsymbol{v}_3 = \begin{bmatrix} 0 \\ \sqrt{3}/2 \\ -\sqrt{3}/2 \end{bmatrix}.$$

(a) (5 Points) Determine $\theta_{12} = \angle(\boldsymbol{v}_1, \boldsymbol{v}_2)$, the angle between vectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$.

(b) (5 Points) Determine $\mu_{\boldsymbol{v}_3} = \mathsf{avg}(\boldsymbol{v}_3)$, the mean of vector $\boldsymbol{v}_3$.

Recall: The mean of a vector $\boldsymbol{y} \in \mathbb{R}^m$ is the arithmetic average of its components—namely,

$$\mu_y = \mathsf{avg}(\boldsymbol{y}) = \frac{y_1 + \cdots + y_m}{m} = \frac{1}{m}\boldsymbol{y}^\mathsf{T}\,\mathbf{1}.$$

(c) (5 Points) Determine $\langle \boldsymbol{v}_2, \boldsymbol{v}_3 \rangle$, the inner product of vectors $\boldsymbol{v}_2$ and $\boldsymbol{v}_3$.

(d) (10 Points) Explain why the set of vectors $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ forms a basis in $\mathbb{R}^3$.

(e) (20 Points) Consider the vector

$$\boldsymbol{x} = \begin{bmatrix} 1/2 \\ -1 \\ 1/2 \end{bmatrix}.$$

(i) (15 Points) Express $\boldsymbol{x}$ as a linear combination of the vectors $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, and $\boldsymbol{v}_3$. That is, determine the coefficients α_1, α_2, and α_3 in the expansion

$$\boldsymbol{x} = \alpha_1\boldsymbol{v}_1 + \alpha_2\boldsymbol{v}_2 + \alpha_3\boldsymbol{v}_3.$$

(ii) (5 Points) Are the values of α_1, α_2, and α_3 that you found unique? If so, explain why. If not, provide another set of coefficients β_1, β_2, and β_3 such that

$$\boldsymbol{x} = \beta_1\boldsymbol{v}_1 + \beta_2\boldsymbol{v}_2 + \beta_3\boldsymbol{v}_3,$$

where $\beta_k \neq \alpha_k$ for at least some $k \in \{1, 2, 3\}$.

MT1.2 (45 Points)

Let $\mathcal{P}_n = \mathsf{span}(1, t, \ldots, t^n)$ denote a real-valued vector space of polynomials of degree less than, or equal to, n, where n is a nonnegative integer and $t \in \mathbb{R}$. A generic polynomial in $\mathcal{P}_n$ can be expressed as follows:

$$p(t) = \sum_{i=0}^{n} p_i\, t^i = \underbrace{\begin{bmatrix} 1 & t & \cdots & t^n \end{bmatrix}}_{\boldsymbol{f}^\mathsf{T}(t)} \underbrace{\begin{bmatrix} p_0 \\ p_1 \\ \vdots \\ p_n \end{bmatrix}}_{\boldsymbol{p}} = \boldsymbol{f}^\mathsf{T}(t)\boldsymbol{p},$$

where $\boldsymbol{f}(t) \in \mathbb{R}^{n+1}$ denotes the vector of monomials (you can think of it as a vector-valued function of t), $\boldsymbol{p} \in \mathbb{R}^{n+1}$ denotes the vector of the coefficients, and $^\mathsf{T}$ denotes transpose.

(a) (5 Points) Determine $\mathsf{dim}\, \mathcal{P}_n$, the dimension of the vector space $\mathcal{P}_n$.

(b) (26 Points) Define $\mathcal{V} \subseteq \mathcal{P}_n$ as the subset of all polynomials in $\mathcal{P}_n$ that have $t = 0$ as a root. That is,

$$\mathcal{V} = \left\{ p(t) = \sum_{i=0}^{n} p_i\, t^i \,\middle|\, p(0) = 0, p_i \in \mathbb{R}, i = 0, \ldots, n \right\}.$$

(i) (12 Points) Explain why $\mathcal{V}$ is a subspace of $\mathcal{P}_n$.

(ii) (10 Points) Determine a basis for $\mathcal{V}$.

(iii) (4 Points) Determine $\dim \mathcal{V}$, the dimension of $\mathcal{V}$.

Explain your answer in a brief, yet clear and convincing manner.

You should be able to solve this part *even if* you're unsure of your solution to part (ii).

(c) (14 Points) Define $\mathcal{W} \subseteq \mathcal{P}_n$ as the subset of all polynomials in $\mathcal{P}_n$ that have $t = 1$ as a root. That is,

$$\mathcal{W} = \left\{ p(t) = \sum_{i=0}^{n} p_i\, t^i \,\middle|\, p(1) = 0, p_i \in \mathbb{R}, i = 0, \ldots, n \right\}.$$

(i) (4 Points) Determine $\dim \mathcal{W}$, the dimension of $\mathcal{W}$.

(ii) (10 Points) Determine a basis for $\mathcal{W}$. Explain your answer in a brief, yet clear and convincing manner.

MT1.3 (40 Points) Consider the vector $\boldsymbol{a} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ in $\mathbb{R}^2$.

(a) (25 Points) Let's look at the subset S of $\mathbb{R}^2$ defined by

$$\mathsf{S} = \left\{ \boldsymbol{x} \in \mathbb{R}^2 \middle| \langle \boldsymbol{a}, \boldsymbol{x} \rangle = 0 \right\}.$$

(i) (5 Points) Describe, in simple words, the vectors $\boldsymbol{x}$ that form the set S.

(ii) (10 Points) Provide a single, well-labeled plot of the vector $\boldsymbol{a}$ and the set S in $\mathbb{R}^2$. Use the standard orthogonal coordinate axes in $\mathbb{R}^2$.

(iii) (10 Points) Is S a subspace of $\mathbb{R}^2$?

If you claim that S is a subspace, prove it.

If you claim that S is <u>not</u> a subspace, show that it fails at least one property of a subspace.

(b) (15 Points) Now let's look at the subset V of $\mathbb{R}^2$ defined by

$$\mathsf{V} = \left\{ \boldsymbol{x} \in \mathbb{R}^2 \middle| \boldsymbol{a}^{\mathsf{T}} \boldsymbol{x} \le 0 \right\}.$$

(i) (5 Points) On a well-labeled plot—using the standard orthogonal coordinate axes in $\mathbb{R}^2$—shade the area corresponding to V.

(ii) (10 Points) Is V a subspace of $\mathbb{R}^2$?

If you claim that V is a subspace, prove it.

If you claim that V is <u>not</u> a subspace, show that it fails at least one property of a subspace.

MT1.4 (35 Points) Consider the following set of twelve vectors in $\mathbb{R}^2$:

$$\boldsymbol{x}_k = \begin{bmatrix} \cos\left(\frac{\pi}{6}k\right) \\ \sin\left(\frac{\pi}{6}k\right) \end{bmatrix}, \qquad \text{for} \qquad k = 0, 1, \ldots, 11.$$

In what follows, you may or may not find it useful to know that

$$\begin{aligned} \cos\left(\frac{\pi}{6}\right) &= \frac{\sqrt{3}}{2} \\ \sin\left(\frac{\pi}{6}\right) &= \frac{1}{2} \\ \cos^2\alpha + \sin^2\alpha &= 1 \\ \cos(\pi + \alpha) &= -\cos\alpha \\ \sin(\pi + \alpha) &= -\sin\alpha. \end{aligned}$$

(a) (10 Points) Determine $||\boldsymbol{x}_k||$, the Euclidean norm (i.e., 2-norm) of $\boldsymbol{x}_k$. Does your expression depend on k? Explain why.

(b) (10 Points) Draw each of the two vectors $\boldsymbol{x}_1$ and $\boldsymbol{x}_7$ on the same coordinate plane defined by the two standard orthogonal axes.

(c) (15 Points) Determine the vector

$$\boldsymbol{v} = \sum_{\substack{k=0 \\ k \neq 6}}^{11} \boldsymbol{x}_k.$$

Your expression for $\boldsymbol{v}$ must be in closed form—not, for example, in terms of a sum.

Hint: First determine the vector

$$\boldsymbol{w} = \sum_{k=0}^{11} \boldsymbol{x}_k,$$

and then infer the vector $\boldsymbol{v}$ from $\boldsymbol{w}$.

MT1.5 (20 Points) Consider the following two vectors in $\mathbb{R}^2_{\geq 0}$:

$$\boldsymbol{v} = \begin{bmatrix} x \\ y \end{bmatrix} \qquad \boldsymbol{w} = \begin{bmatrix} y \\ x \end{bmatrix}.$$

We denote by $\mathbb{R}_{\geq 0}$ the set of all nonnegative real numbers—that is,

$$\mathbb{R}_{\geq 0} = \{x \in \mathbb{R} | \, x \geq 0\}.$$

And we denote by $\mathbb{R}^2_{\geq 0}$ the set of all vectors in $\mathbb{R}^2$ that have nonnegative components.

Accordingly, $x \geq 0$ and $y \geq 0$ above.

(a) (15 Points) Show that

$$xy \leq \frac{x^2 + y^2}{2}.$$

Hint: Study the inner product $\langle \boldsymbol{v}, \boldsymbol{w} \rangle$ and make judicious use of the Cauchy-Schwarz Inequality.

(b) (5 Points) Show that for any $a, b \geq 0$, the following inequality holds:

$$\sqrt{ab} \leq \frac{a + b}{2}.$$