EECS 16A: Foundations of Signals, Dynamical Systems, and Information Processing — Midterm 1

Department of Electrical Engineering and Computer Sciences

University of California, Berkeley — 11 February 2025

FIRST Name: Ortha LAST Name: Vektir SID (All Digits): 123456789

- **(5 Points)** On *every* page, print legibly your name and ALL digits of your SID. For every page on which you do not write your name and SID, you forfeit a point, up to the maximum 5 points.
- **(10 Points) (Pledge of Academic Integrity)** Hand-copy, sign, and date the single-line text (which begins with *I have read, ...*) of the Pledge of Academic Integrity on page 3 of this document. Your solutions will *not* be evaluated without this.
- **Urgent Contact with the Teaching Staff:** In case of an urgent matter, raise your hand if in-person, or send an email to `eecs16a@berkeley.edu` if online.
- **This document consists of pages numbered 1 through 18.** Verify that your copy of the exam is free of anomalies, and contains all of the specified number of pages. If you find a defect in your copy, contact the teaching staff immediately.
- This exam is designed to be completed within 70 minutes. However, you may use up to 80 minutes total—*in one sitting*—to tackle the exam.

  The exam starts at 8:10 pm California time. Your allotted window begins with respect to this start time. Students who have official accommodations of 1.5× and 2× time windows have 120 and 160 minutes, respectively.
- **This exam is closed book.** You may not use or access, or cause to be used or accessed, any reference in print or electronic form at any time during the exam, except one double-sided 8.5" × 11" sheet of handwritten, original notes having no appendage.

  Collaboration is not permitted.

  Computing, communication, and other electronic devices (except dedicated timekeepers) must be turned off.

  Scratch paper will be provided to you; ask for more if you run out. You may not use your own scratch paper.
- Please write neatly and legibly, because *if we can't read it, we can't evaluate it.*
- For each problem, limit your work to the space provided specifically for that problem. *No other work will be considered. For example, we will not evaluate scratch work. No exceptions.*
- Unless explicitly waived by the specific wording of a problem, you must explain your responses (and reasoning) succinctly, but clearly and convincingly.
- In some parts of a problem, we may ask you to establish a certain result—for example, "show this" or "prove that." Even if you're unable to establish the result that we ask of you, you may still take that result for granted—and use it in any subsequent part of the problem.

- If we ask you to provide a "reasonably simple expression" for something, by default we expect your expression to be in closed form—one *not* involving a sum $\sum$ or an integral $\int$—*unless* we explicitly tell you otherwise.
- Noncompliance with these or other instructions from the teaching staff—*including, for example, commencing work prematurely, or continuing it beyond the allocated time window*—is a serious violation of the Code of Student Conduct.

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Pledge of Academic Integrity

By my honor, I affirm that

(1) this document—which I have produced for the evaluation of my performance—reflects my original, bona fide work, and that I have neither provided to, nor received from, anyone excessive or unreasonable assistance that produces unfair advantage for me or for any of my peers;

(2) as a member of the UC Berkeley community, I have acted with honesty, integrity, respect for others, and professional responsibility—and in a manner consistent with the letter and intent of the campus Code of Student Conduct;

(3) I have not violated—nor aided or abetted anyone else to violate—the instructions for this exam given by the course staff, including, but not limited to, those on the cover page of this document; and

(4) More generally, I have not committed any act that violates—nor aided or abetted anyone else to violate—UC Berkeley, state, or Federal regulations, during this exam.

(10 Points) In the space below, hand-write the following sentence, verbatim. Then write your name in legible letters, sign, include your full SID, and date before submitting your work:

I have read, I understand, and I commit to adhere to the letter and spirit of the pledge above.

I have read, I understand, and I commit to the letter and spirit of the pledge above.

Full Name: Ortha Vektir Signature:

Date: 11 Feb 2025 Student ID: 123456789

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Potentially Useful Facts That You May Use Without the Need to Prove Them:

- **Inner Product:** For every $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$, we define $\langle \boldsymbol{x}, \boldsymbol{y} \rangle \stackrel{\triangle}{=} \boldsymbol{x}^\intercal \boldsymbol{y} = \displaystyle\sum_{k=1}^{n} x_k \, y_k$.

- **Cauchy-Schwarz Inequality:** For all elements x and y in a vector space $\mathcal{V}$,
$$\left|\langle x, y \rangle\right| \leq ||x|| \, ||y||.$$

- **Triangle Inequality:** For all elements x and y in a vector space $\mathcal{V}$,
$$||x + y|| \leq ||x|| + ||y||.$$

- **Geometric Sum Formula** For all integers M and N, where $M \leq N$,
$$\sum_{\ell=M}^{N} \alpha^\ell = \begin{cases} \dfrac{\alpha^{N+1} - \alpha^M}{\alpha - 1} & \text{if } \alpha \neq 1 \\ N - M + 1 & \text{if } \alpha = 1. \end{cases}$$

- **Angle Between Vectors:** The angle θ between two *nonzero* elements x and y in a *real* vector space satisfies
$$\theta = \arccos \frac{\langle x, y \rangle}{||x|| \, ||y||}, \qquad \text{and} \qquad \cos\theta = \frac{\langle x, y \rangle}{||x|| \, ||y||}.$$
Whether in a real or complex vector space, if $\langle x, y \rangle = 0$, we say x and y are orthogonal, and we denote this by $x \perp y$.

- **Polynomials**
  - Any *nonzero* polynomial $p(t) = \displaystyle\sum_{k=0}^{n} a_k t^k$ in a real variable t, having real coefficients a_k, of degree $n \geq 0$, has exactly n roots, inclusive of multiplicity (i.e., root repetition)—real or complex.
  - Any polynomial $p(t) = \displaystyle\sum_{k=0}^{n} a_k t^k$ of degree $n \geq 0$ is infinitely differentiable—that is, it has derivatives of all orders.

- **Some Trigonometric Values**:
$$\cos\left(\frac{\pi}{6}\right) = \sin\left(\frac{\pi}{3}\right) = \sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2} \qquad \sin\left(\frac{\pi}{6}\right) = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2} \qquad \cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2}.$$
$$\sin\left(\frac{\pi}{2}\right) = 1 \qquad \cos\left(\frac{\pi}{2}\right) = \cos\left(\frac{3\pi}{2}\right) = 0 \qquad \sin\left(\frac{3\pi}{2}\right) = -1.$$

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MT1.1 (25 Points) Nearest-Neighbor Classification

Suppose

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$$

denotes a feature vector for Movie X. Based on the two features x_1 and x_2, we want to classify this movie as type *Action*, *Behind-the-Scenes*, or *Crime*.

The prototype feature vectors

$$\boldsymbol{a} = \begin{bmatrix} -1/2 \\ 0 \end{bmatrix}, \qquad \boldsymbol{b} = \begin{bmatrix} 1/2 \\ 0 \end{bmatrix}, \qquad \text{and} \qquad \boldsymbol{c} = \begin{bmatrix} 0 \\ \sqrt{3}/2 \end{bmatrix}$$

represent *Action*, *Behind-the-Scenes*, and *Crime* movie genres, respectively.

To assign Movie X to one of the three genres, we measure the Euclidean distances (i.e., in 2-norm) between its feature vector $\boldsymbol{x}$ and each of the vectors $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$.

The distance to whichever prototype is shortest, we call that prototype the nearest neighbor of $\boldsymbol{x}$ and assign Movie X to the genre corresponding to that nearest neighbor.

For example, if $\boldsymbol{x}$ is no farther from $\boldsymbol{a}$ than it is from each of $\boldsymbol{b}$ and $\boldsymbol{c}$, we classify X as an *Action* movie. In mathematical terms, if

$$||\boldsymbol{x} - \boldsymbol{a}|| \leq ||\boldsymbol{x} - \boldsymbol{b}||$$

and

$$||\boldsymbol{x} - \boldsymbol{a}|| \leq ||\boldsymbol{x} - \boldsymbol{c}||,$$

we classify X an *Action* movie.

The figure on the next page shows the locations of the three prototype vectors $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$.

Determine the regions in $\mathbb{R}^2$ where Movie X should be assigned to each of the three categories.

You need not write detailed equations for the decision boundaries. But you *must* explain how you obtain those boundaries, and you *must* identify and label the coordinates of all the important points, such as where the bisectors meet or where each bisector intersects a side of an important triangle.

Note: *Only work shown on the next page will be evaluated.*

Hint: For an equilateral triangle (one whose three sides are equal in length), the *circumcenter*—that is, the point where the three bisectors meet—is 1/3 of the way from each side, along the bisector, toward the opposing vertex.

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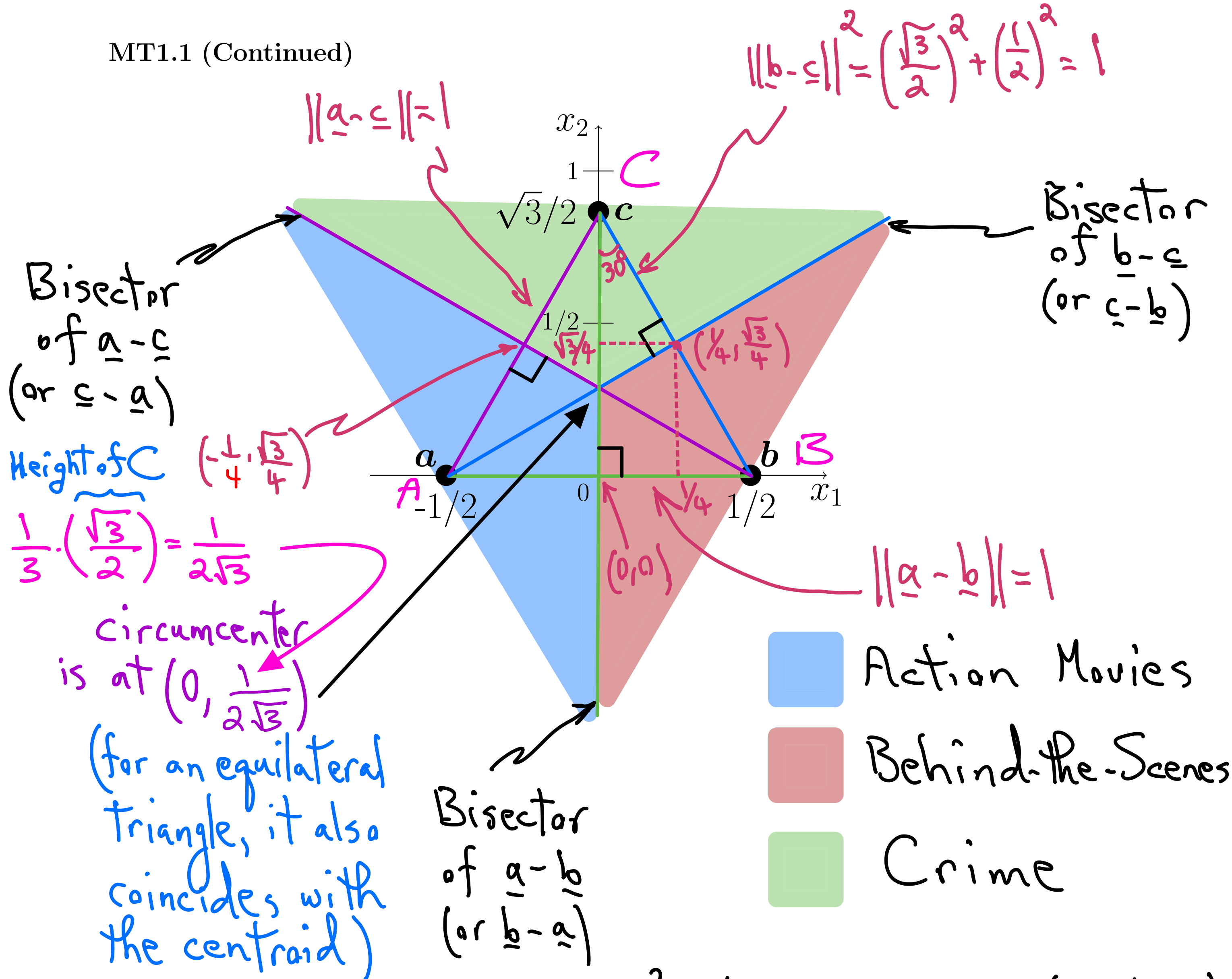
MT1.1 (Continued)



Bisector of $\underline{b}-\underline{c}$ splits $\mathbb{R}^2$ into two regions (halfplanes). The lower part consists of points closer to $\underline{b}$, whereas the upper part consists of points closer to $\underline{c}$.

The other two bisectors act in a similar manner. The region is the intersection of all halfplanes closer to $\underline{a}$. The is the intersection of halfplanes closer to $\underline{b}$. And is the intersection of halfplanes closer to $\underline{c}$.

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MT1.2 (30 Points) Polynomials

Let $\mathcal{P}_2$ denote a real-valued vector space of polynomials of degree less than, or equal to, 2. One way to think of $\mathcal{P}_2$ is as the set constructed from all real linear combinations of

$$\varphi_0(t) = 1, \quad \varphi_1(t) = t, \quad \text{and} \quad \varphi_2(t) = t^2, \quad \forall t \in \mathbb{R}.$$

An arbitrary polynomial in $\mathcal{P}_2$ can be expressed as follows:

$$p(t) = p_0 + p_1 t + p_2 t^2 = \underbrace{\begin{bmatrix} 1 & t & t^2 \end{bmatrix}}_{\boldsymbol{f}(t)^\top} \underbrace{\begin{bmatrix} p_0 \\ p_1 \\ p_2 \end{bmatrix}}_{\boldsymbol{p}} = \boldsymbol{f}(t)^\top \boldsymbol{p},$$

where $\boldsymbol{f}(t) \in \mathbb{R}^3$ denotes the vector of monomials (you can think of $\boldsymbol{f}$ as a vector-valued function of the continuous variable t), $\boldsymbol{p} \in \mathbb{R}^3$ denotes the vector of the coefficients, and $^\top$ denotes transpose.

Consider the set

$$\mathcal{A} = \{\psi_1(t), \psi_2(t)\},$$

where $\psi_1(t) = 1 + t$ and $\psi_2(t) = (1+t)^2$ for all real t.

We can think of the span of $\mathcal{A}$ as

$$\mathsf{span}(\mathcal{A}) = \left\{ q(t) = \alpha(1+t) + \beta(1+t)^2 \,\middle|\, \forall \alpha, \beta \in \mathbb{R} \right\}.$$

In particular, suppose

$$q(t) = q_0 + q_1 t + q_2 t^2$$

is an arbitrary polynomial in $\mathsf{span}(\mathcal{A})$.

(a) (10 Points) Determine the coefficients q_0, q_1, and q_2 in terms of α and β.

$$q(t) = \alpha(1+t) + \beta(1+t)^2$$

$$= \alpha + \alpha t + \beta(1 + 2t + t^2)$$

$$q(t) = \underbrace{(\alpha+\beta)}_{q_0} + \underbrace{(\alpha+2\beta)}_{q_1} t + \underbrace{\beta}_{q_2} t^2$$

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MT1.2 (Continued)

(b) (10 Points) Determine one of the roots of $q(t)$ numerically. This root is common among all polynomials in $\mathsf{span}(\mathcal{A})$.

Method I: $q(t) = \alpha(1+t) + \beta(1+t)^2 = (1+t)\left[\alpha + \beta(1+t)\right] = 0 \Longrightarrow$ $t=-1$ is a root. A linear combo of $1+t$ and $(1+t)^2$ — each of which has $t=-1$ as a root — must have $t=-1$ as a root.

Method II: A quadratic equation $q(t) = q_0 + q_1 t + q_2 t^2 = 0$ has $t=-1$ as a root if, and only if, $q_0 - q_1 + q_2 = 0$ (i.e., $q_0 + q_2 = q_1$)

From (a), we know $q_0 = \alpha+\beta$, $q_2 = \beta$, and $q_1 = \alpha + 2\beta = q_0 + q_2 \Rightarrow t=-1$ is a root.

(c) (10 Points) Does $\mathcal{A}$ form a basis for $\mathcal{P}_2$? If you answer in the affirmative, then prove it. If you answer in the negative, then find one polynomial in $\mathcal{P}_2$ that is *not* in the span of $\mathcal{A}$, and determine $\mathsf{dim}(\mathsf{span}(\mathcal{A}))$.

No! The set $\mathcal{A}$ is not a basis for $\mathcal{P}_2$. Clearly, $\mathcal{P}_2$ includes polynomials (quadratics) neither of whose roots is -1. An example is $p(t) = a(t-1)(t+2)$ $\forall a \neq 0$. This polynomial cannot be expressed as a linear combo of $1+t$ and $(1+t)^2$

Alternative View: There's a one-to-one correspondence between every quadratic $q(t) = q_0 + q_1 t + q_2 t^2$ and vectors in $\mathbb{R}^3$. In particular, the vector $\underline{q} = [q_0 \;\; q_1 \;\; q_2]^T$ completely identifies the quadratic, and vice versa. From Part (a) we know

$$\underline{q} = \begin{bmatrix} \alpha+\beta \\ \alpha+2\beta \\ \beta \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$ In other words $\underline{q} \in \mathrm{span}\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}\right)$

Even though $\left\{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}\right\}$ is a linearly independent set, it doesn't have the requisite number of vectors to span $\mathbb{R}^3$, so it isn't a basis.

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MT1.3 (30 Points) Of Sines and Cosines

In this problem we explore some vector-space properties of trigonometric functions.

(a) (10 Points) Consider the set $\mathcal{A} = \{\varphi_0(t), \varphi_1(t), \varphi_2(t)\}$, where

$$\forall t \in \mathbb{R}, \qquad \varphi_0(t) = 1, \qquad \varphi_1(t) = \cos t, \quad \text{and} \quad \varphi_2(t) = \sin t.$$

Prove that $\mathcal{A}$ is linearly independent. To do this, you must set an arbitrary linear combination

$$\alpha\,\varphi_0(t) + \beta\,\varphi_1(t) + \gamma\,\varphi_2(t) = 0 \quad \text{for all } t \in \mathbb{R},$$

and then show that the coefficients α, β, and γ must all be zero.

$$\alpha\,\varphi_0(t) + \beta\,\varphi_1(t) + \gamma\,\varphi_2(t) = 0 \quad \forall t \in \mathbb{R}$$

$$\alpha\,\underbrace{1}_{} + \beta \cos t + \gamma \sin t = 0 \qquad (*)$$

Since $(*)$ must hold for all $t \in \mathbb{R}$, we make a judicious selection of values of t that gives us three equations in three unknowns:

$$t = 0 \Longrightarrow \cos t\big|_{t=0} = 1 \text{ and } \sin t\big|_{t=0} = 0 \Rightarrow \alpha + \beta = 0 \Rightarrow \beta = -\alpha$$

$$t = \frac{\pi}{2} \Longrightarrow \cos t\big|_{t=\pi/2} = 0 \text{ and } \sin t\big|_{t=\frac{\pi}{2}} = 1 \Rightarrow \alpha + \gamma = 0 \Rightarrow \gamma = -\alpha$$

$$t = \frac{\pi}{4} \rightarrow \cos t\big|_{t=\pi/4} = \frac{1}{\sqrt{2}} \text{ and } \sin t\big|_{t=\pi/4} = \frac{1}{\sqrt{2}} \Rightarrow \alpha + \frac{1}{\sqrt{2}}\beta + \frac{1}{\sqrt{2}}\gamma = 0$$

So, $\alpha - \frac{1}{\sqrt{2}}\alpha - \frac{1}{\sqrt{2}}\alpha = 0 \Rightarrow \alpha = 0 \Rightarrow \beta = -\alpha = 0$ & $\gamma = -\alpha = 0$

Since all three coefficients are zero, the set $\{1, \cos t, \sin t\}$ must be linearly independent.

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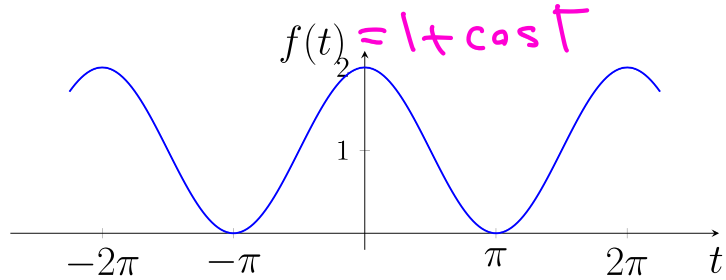
MT1.3 (Continued)

(b) (20 Points) Consider the set $\mathcal{B} = \{1, \cos t\}$, where 1 and cosine are functions defined over all real t.

(i) (10 Points) The function f shown below is in the span of $\mathcal{B}$.



$= 1 + \cos t$

Determine the coefficients α and β such that $f(t) = \alpha + \beta \cos t$.

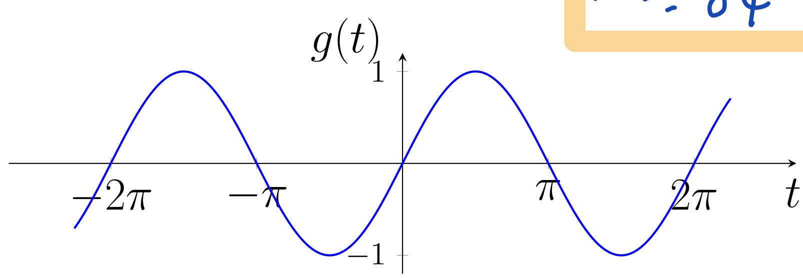
$f(t) = 1 + \cos t$ which matches the plot.

Therefore $\alpha = 1$, $\beta = 1$

(ii) (10 Points) Consider the function g shown below:

No! $g \notin \operatorname{span}\{1, \cos t\}$

$g(t) = \sin t$



Is g in the span of $\mathcal{B}$? If yes, provide the coefficients in the linear combination $g(t) = \lambda + \mu \cos t$. If you assert that g is *not* in the span of $\mathcal{B}$, explain why.

Method I: From Part (a) we know $\{1, \cos t, \sin t\}$ is linearly indep. So, none of its elements can be expressed as a linear combo of the others. Hence $g(t) = \sin t$ is not a lin. combo of $1, \cos t$.

Method II: Each of the functions $\varphi_1(t) = 1 \ \forall t$ and $\varphi_2(t) = \cos t \ \forall t$ is even — that is $\varphi_1(-t) = \varphi_1(t)$ and $\varphi_2(-t) = \varphi_2(t)$. Every function, say h, obtained from φ_0 & φ_1 by a linear combo is even. But g is odd.

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MT1.4 (40 Points) On the Basis of Independence

Consider the following six vectors in $\mathbb{R}^5$:

$$\boldsymbol{u} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \qquad \boldsymbol{v} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \qquad \boldsymbol{w} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix},$$

$$\boldsymbol{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \qquad \boldsymbol{y} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 3 \\ 3 \end{bmatrix}, \qquad \boldsymbol{z} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \\ 3 \end{bmatrix}$$

(a) (10 Points) Show that the set $\{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\}$ is linearly independent.

$\underline{u} = \begin{bmatrix} \underline{e}_1 \\ \mathbb{1} \end{bmatrix}$ $\underline{v} = \begin{bmatrix} \underline{e}_2 \\ \mathbb{1} \end{bmatrix}$ $\underline{w} = \begin{bmatrix} \underline{e}_3 \\ \mathbb{1} \end{bmatrix}$, where $\underline{u}, \underline{v}, \underline{w} \in \mathbb{R}^5$

$\underline{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \underline{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \underline{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \in \mathbb{R}^3$

$\mathbb{1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \in \mathbb{R}^2$

Let

$\alpha \underline{u} + \beta \underline{v} + \gamma \underline{w} = \underline{0}$. We have

$\begin{bmatrix} \alpha \underline{e}_1 + \beta \underline{e}_2 + \gamma \underline{e}_3 \\ (\alpha + \beta + \gamma) \mathbb{1} \end{bmatrix} = \begin{bmatrix} \underline{0}_3 \\ \underline{0}_2 \end{bmatrix} \Rightarrow \alpha \underline{e}_1 + \beta \underline{e}_2 + \gamma \underline{e}_3 = 0 \Rightarrow$

$\alpha = \beta = \gamma = 0$ since

$\underline{e}_1, \underline{e}_2, \underline{e}_3$ are indep.

$\Rightarrow \underline{u}, \underline{v}, \underline{w}$ are indep.

MT1.4 (Continued)

(b) (10 Points) Is the set $\{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\}$ a basis for $\mathbb{R}^5$? If your answer is in the affirmative, prove that every vector in $\mathbb{R}^5$ can be expressed as a linear combination of the vectors in the set. If you claim the set is not a basis, find one vector in $\mathbb{R}^5$ that is *not* in the span of the set.

No. Even though $\{\underline{u}, \underline{v}, \underline{w}\}$ is independent, the set is missing two more independent vectors needed for a basis in $\mathbb{R}^5$.

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MT1.4 (Continued)

(c) (10 Points) Is the set $\{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}, \boldsymbol{x}, \boldsymbol{y}\}$ a basis for $\mathbb{R}^5$? Explain your reasoning. No!

Method I:

The vector $\underline{y}$ can be expressed as

$$\underline{y} = \underline{u} + \underline{w} + \underline{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 3 \\ 3 \end{bmatrix} \Longrightarrow$$

$\{\underline{u}, \underline{v}, \underline{w}, \underline{x}, \underline{y}\}$ is a linearly dependent set.
Even though the set has the correct number 5 of elements, it's a linearly depentent set. Accordingly, it can't serve as a basis.

Method II: Gaussian elimination!?

No, you didn't!

Say you didn't!

MT1.4 (Continued)

(d) (10 Points) Is the set $\{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}, \boldsymbol{x}, \boldsymbol{z}\}$ a basis for $\mathbb{R}^5$? Explain your reasoning.

Yes.

We already know that $\{\underline{u}, \underline{v}, \underline{w}\}$ is linearly independent.

The vector $\underline{x}$ in block form is $\underline{x} = \begin{bmatrix} \underline{a} \\ \underline{b} \end{bmatrix} = \begin{bmatrix} \underline{0}_3 \\ \mathbb{1}_2 \end{bmatrix}$.

But $\underline{a} = \underline{0} \in \mathbb{R}^3$ can only be obtained from the corresponding portions $\underline{e}_1$, $\underline{e}_2$, and $\underline{e}_3$ of $\underline{u}, \underline{v}$, and $\underline{w}$ through the combination $0\underline{e}_1 + 0\underline{e}_2 + 0\underline{e}_3$. But that linear combination of $\underline{u}, \underline{v}$, and $\underline{w}$ cannot produce the lower block $\underline{b}$ of vector $\underline{x}$. So, we conclude that $\{\underline{u}, \underline{v}, \underline{w}, \underline{x}\}$ is linearly independent.

As for $\underline{z} = \begin{bmatrix} \underline{c} \\ \underline{d} \end{bmatrix} = \begin{bmatrix} \mathbb{1}_3 \\ \begin{bmatrix} 2 \\ 3 \end{bmatrix} \end{bmatrix}$, where $\underline{c} = \mathbb{1} \in \mathbb{R}^3$ and $\underline{d} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, we note that every linear combination of $\underline{u}, \underline{v}, \underline{w}$, and $\underline{x}$ produces identical values in the last two entries — that is,

$$\alpha\underline{u} + \beta\underline{v} + \gamma\underline{w} + \lambda\underline{x} = \begin{bmatrix} \alpha \\ \beta \\ \gamma \\ \alpha+\beta+\gamma+\lambda \\ \alpha+\beta+\gamma+\lambda \end{bmatrix} \leftarrow \text{identical.}$$

But $\underline{z}$'s last two entries are 2 and 3 — nonidentical.

So $\underline{z} \notin \text{span}(\underline{u}, \underline{v}, \underline{w}, \underline{x}) \implies \{\underline{u}, \underline{v}, \underline{w}, \underline{x}, \underline{z}\}$ is linearly independent. It also has five vectors, so it can serve as a basis in $\mathbb{R}^5$.

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MT1.5 (30 Points) Cauchy & Schwarz Go to AM-GM in Las Vegas!

Consider n *nonnegative* real numbers $x_1, \ldots, x_n$. The arithmetic mean of these numbers is

$$\mathrm{AM} \triangleq \frac{x_1 + \cdots + x_n}{n} = \frac{1}{n}\sum_{k=1}^{n} x_k.$$

The geometric mean of these same numbers is

$$\mathrm{GM} \triangleq \sqrt[n]{x_1 \cdots x_n} = \sqrt[n]{\prod_{k=1}^{n} x_k}.$$

The *AM-GM Inequality* states that the geometric mean cannot exceed the arithmetic mean:

$$\sqrt[n]{x_1 \cdots x_n} \leq \frac{x_1 + \cdots + x_n}{n}.$$

Let's derive the AM-GM Inequality for the cases $n = 2$ and $n = 3$.

(a) (10 Points) Consider a pair of nonnegative real values x and y. Show that $\sqrt{xy} \leq \dfrac{x+y}{2}$.

Hint: Two options: **Method I:** Exploit the fact that $(x-y)^2 \geq 0$. **Method II:** Let $\boldsymbol{u} = \begin{bmatrix} \sqrt{x} \\ \sqrt{y} \end{bmatrix}$. Choose a vector $\boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ in a judicious manner, so that a simple application of the Cauchy-Schwarz inequality to $\boldsymbol{u}$ and $\boldsymbol{v}$ takes you immediately to the desired result.

Method I:

$$\sqrt{xy} \leq \frac{x+y}{2} \underset{x,y\geq 0}{\iff} xy \leq \frac{(x+y)^2}{4} \iff (x+y)^2 \geq 4xy \iff$$

$$(x+y)^2 - 4xy \geq 0 \iff x^2 + 2xy + y^2 - 4xy = x^2 - 2xy + y^2 \geq 0$$

$\iff (x-y)^2 \geq 0$, which is true.

Method II: Let $\underline{u} = \begin{bmatrix} \sqrt{x} \\ \sqrt{y} \end{bmatrix}$ & $\underline{v} = \begin{bmatrix} \sqrt{y} \\ \sqrt{x} \end{bmatrix}$. The Cauchy-Schwarz Inequality states that $\langle \underline{u}, \underline{v} \rangle \leq \|\underline{u}\|\,\|\underline{v}\|$.

Cauchy-Schwarz

But $\langle \underline{u}, \underline{v} \rangle = \left\langle \begin{bmatrix} \sqrt{x} \\ \sqrt{y} \end{bmatrix}, \begin{bmatrix} \sqrt{y} \\ \sqrt{x} \end{bmatrix} \right\rangle = \sqrt{xy} + \sqrt{yx} = 2\sqrt{xy}$

$\|\underline{u}\| = \|\underline{v}\| = \sqrt{(\sqrt{x})^2 + (\sqrt{y})^2} = \sqrt{x+y} \implies \|\underline{u}\|\,\|\underline{v}\| = x+y$

$\implies 2\sqrt{xy} \leq x+y$

$\Downarrow$

$\sqrt{xy} \leq \dfrac{x+y}{2}$

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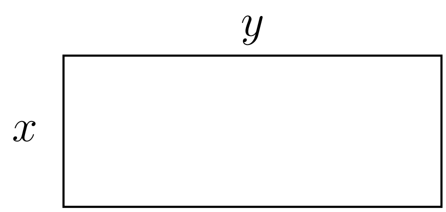
MT1.5 (Continued)

(b) (10 Points) Denote the height of a rectangle by $x > 0$ and its width by $y > 0$.



Consider the set of all rectangles whose perimeters are $2(x+y) = 4$ meters. Determine numerical values for the width x and height y of the rectangle that has maximum area.

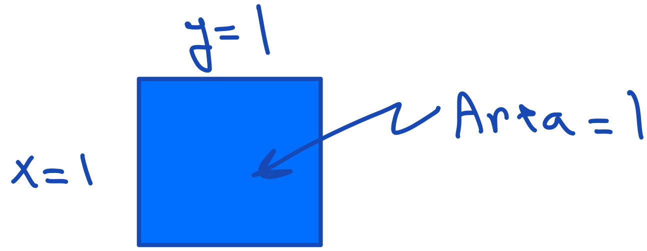
GM Inequality: $\sqrt{xy} \leq \frac{x+y}{2} \implies 4\sqrt{xy} \leq 2(x+y)$

Equality in the GM Inequality occurs if, and only if, equality in the Cauchy-Schwarz Inequality occurs (or, if, and only if, equality in $(x-y)^2 \geq 0$ occurs) and that condition is $x=y$.

So, the largest-area rectangle having height x and width y is actually a square — with $x=y$.

Our Square has perimeter $\left.\begin{matrix} 2(x+y)=4 \\ x=y \end{matrix}\right\} \implies$

$x=y=1$.



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MT1.5 (Continued)

(c) (10 Points) Prove the AM-GM Inequality for the case $n = 3$—that is, for $x, y, z \geq 0$, prove that

$$\sqrt[3]{xyz} \leq \frac{x+y+z}{3}.$$

Hint: Let $\boldsymbol{a} = \begin{bmatrix} u \\ v \\ w \end{bmatrix}$ and $\boldsymbol{b} = \begin{bmatrix} v \\ w \\ u \end{bmatrix}$. Apply the Cauchy-Schwarz Inequality to $\boldsymbol{a}$ and $\boldsymbol{b}$.
Use the algebraic identity

$$(u+v+w)\left(u^2+v^2+w^2-uv-uw-vw\right) = u^3+v^3+w^3-3uvw.$$

And use the substitutions $u = \sqrt[3]{x}$, $v = \sqrt[3]{y}$, and $w = \sqrt[3]{z}$.

Cauchy-Schwarz $\Rightarrow \langle \underline{a}, \underline{b} \rangle \leq \|\underline{a}\| \|\underline{b}\|$

$\langle \underline{a}, \underline{b} \rangle = uv + uw + vw$

$\|\underline{a}\| = \|\underline{b}\| = \sqrt{u^2+v^2+w^2} \Longrightarrow \|\underline{a}\|\|\underline{b}\| = u^2+v^2+w^2$

$\Rightarrow$

$uv + uw + vw \leq u^2+v^2+w^2 \Longrightarrow$

$u^2+v^2+w^2-uv-uw-vw \geq 0 \Longrightarrow$ Since $u, v, w \geq 0$

$(u+v+w)(u^2+v^2+w^2-uv-uw-vw) = u^3+v^3+w^3-3uvw \geq 0$

$\Rightarrow uvw \leq \dfrac{u^3+v^3+w^3}{3}$

Let $u = \sqrt[3]{x}$, $v = \sqrt[3]{y}$, and $w = \sqrt[3]{z}$

$\Downarrow$ $\Downarrow$ $\downarrow$

$u^3 = x$, $v^3 = y$, and $w^3 = z$

$\Rightarrow \sqrt[3]{xyz} \leq \dfrac{x+y+z}{3}$

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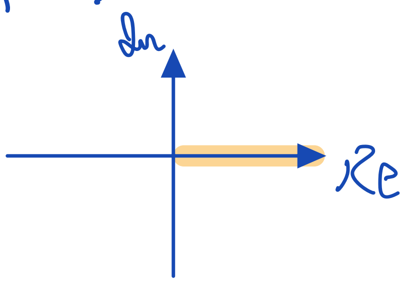
MT1.6 (30 Points) It's Complexicated!

Throughout this problem, i denotes $\sqrt{-1}$, and the Cartesian form of a complex number z is $z = a + ib$, where $a, b \in \mathbb{R}$.

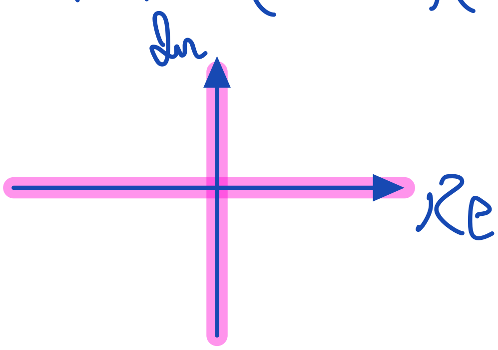
(a) (10 Points) Determine the set of all $z \in \mathbb{C}$ such that $z = |z|$.

$z = |z| \Longrightarrow z \in \mathbb{R}$ and $z \geq 0 \Longrightarrow z$ is a nonnegative real number.



(b) (10 Points) Determine the set of all $z \in \mathbb{C}$ such that $z^2 = (z^*)^2$.

$$z^2 - (z^*)^2 = (z - z^*)(z + z^*) = 0 \Longrightarrow \begin{cases} z = z^* \Longrightarrow z \in \mathbb{R} \\ \text{or} \\ z = -z^* \Longrightarrow z \text{ purely imaginary} \end{cases}$$



(c) (10 Points) Numerically evaluate $\sum_{k=0}^{999} i^k$.

$$\sum_{k=0}^{999} i^k = \frac{i^{999+1} - i^0}{i - 1} = \frac{\cancel{i^{1000}}^{\,1} - 1}{i - 1} = 0$$

1000 is a multiple of 4 $\Longrightarrow i^{1000} = i^{4k} = 1$

$$\boxed{\sum_{k=0}^{999} i^k = 0}$$