**EECS 16A: Foundations of Signals, Dynamical Systems, and Information Processing** Exam 2

Department of Electrical Engineering and Computer Sciences

UNIVERSITY OF CALIFORNIA, BERKELEY 20 March 2025

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- **(5 Points)** On *every* page, print legibly your name and ALL digits of your SID. For every page on which you do not write your name and SID, you forfeit a point, up to the maximum 5 points.

- **(10 Points) (Pledge of Academic Integrity)** Hand-copy, sign, and date the single-line text (which begins with *I have read, . . .*) of the Pledge of Academic Integrity on page 3 of this document. Your solutions will *not* be evaluated without this.

- **Urgent Contact with the Teaching Staff:** In case of an urgent matter, raise your hand if in-person, or send an email to `eecs16a@berkeley.edu` if online.

- **This document consists of pages numbered 1 through 16.** Verify that your copy of the exam is free of anomalies, and contains all of the specified number of pages. If you find a defect in your copy, contact the teaching staff immediately.

- This exam is designed to be completed within 70 minutes. However, you may use up to 80 minutes total—*in one sitting*—to tackle the exam.

  The exam starts at 8:10 pm California time. Your allotted window begins with respect to this start time. Students who have official accommodations of 1.5× and 2× time windows have 120 and 160 minutes, respectively.

- **This exam is closed book.** You may not use or access, or cause to be used or accessed, any reference in print or electronic form at any time during the exam, except two double-sided 8.5" × 11" sheets of handwritten, original notes having no appendage.

  Collaboration is not permitted.

  Computing, communication, and other electronic devices (except dedicated timekeepers) must be turned off.

  Scratch paper will be provided to you; ask for more if you run out. You may not use your own scratch paper.

- Please write neatly and legibly, because *if we can't read it, we can't evaluate it.*

- For each problem, limit your work to the space provided specifically for that problem. *No other work will be considered. For example, we will not evaluate scratch work. No exceptions.*

- Unless explicitly waived by the specific wording of a problem, you must explain your responses (and reasoning) succinctly, but clearly and convincingly.

- In some parts of a problem, we may ask you to establish a certain result—for example, "show this" or "prove that." Even if you're unable to establish the result that we ask of you, you may still take that result for granted—and use it in any subsequent part of the problem.

- If we ask you to provide a "reasonably simple expression" for something, by default we expect your expression to be in closed form—one *not* involving a sum $\sum$ or an integral $\int$—*unless* we explicitly tell you otherwise.
- Noncompliance with these or other instructions from the teaching staff—*including, for example, commencing work prematurely, or continuing it beyond the allocated time window*—is a serious violation of the Code of Student Conduct.

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**Pledge of Academic Integrity**

By my honor, I affirm that

(1) this document—which I have produced for the evaluation of my performance—reflects my original, bona fide work, and that I have neither provided to, nor received from, anyone excessive or unreasonable assistance that produces unfair advantage for me or for any of my peers;

(2) as a member of the UC Berkeley community, I have acted with honesty, integrity, respect for others, and professional responsibility—and in a manner consistent with the letter and intent of the campus Code of Student Conduct;

(3) I have not violated—nor aided or abetted anyone else to violate—the instructions for this exam given by the course staff, including, but not limited to, those on the cover page of this document; and

(4) More generally, I have not committed any act that violates—nor aided or abetted anyone else to violate—UC Berkeley, state, or Federal regulations, during this exam.

**(10 Points)** In the space below, hand-write the following sentence, verbatim. Then write your name in legible letters, sign, include your full SID, and date before submitting your work:

*I have read, I understand, and I commit to adhere to the letter and spirit of the pledge above.*

I have read, I understand, and I commit to adhere to the letter and

spirit of the pledge above.

Full Name: Perpen Deekule

Signature: Perpen Deekule

Date: 20 March 2025

Student ID: 123456789

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**Potentially Useful Facts That You May Use Without the Need to Prove Them:**

- **Angle Between Vectors:** The angle $\theta$ between two *nonzero* elements $x$ and $y$ in a *real* vector space satisfies

$$\theta = \arccos \frac{\langle x, y\rangle}{||x||\,||y||}, \qquad \text{and} \qquad \cos\theta = \frac{\langle x, y\rangle}{||x||\,||y||}.$$

  Whether in a real or complex vector space, if $\langle x, y\rangle = 0$, we say $x$ and $y$ are orthogonal, and we denote this by $x \perp y$.

- **Discrete Fourier Series (DTFS)**: Complex exponential Fourier series synthesis and analysis equations for a periodic discrete-time signal having period $p$:

$$x[n] = \sum_{k=\langle p\rangle} X_k\, e^{ik\omega_0 n} \qquad \longleftrightarrow \qquad X_k = \frac{1}{p}\sum_{n=\langle p\rangle} x[n]\, e^{-ik\omega_0 n}\ ,$$

  where $p = \dfrac{2\pi}{\omega_0}$ and $\langle p\rangle$ denotes a suitable discrete interval of length $p$ (i.e., an interval containing $p$ contiguous integers). For example, $\sum\limits_{k=\langle p\rangle}$ may denote $\sum\limits_{k=0}^{p-1}$ or $\sum\limits_{k=1}^{p}$.

- **Parseval's Identity**: Consider an orthogonal set of vectors $\boldsymbol{\varphi}_k$, $k = 1, \ldots, n$. Suppose $\|\boldsymbol{\varphi}_k\|^2 = c$ for all $k \in \{1, \ldots, n\}$ and some $c > 0$. For any vector $\boldsymbol{x} \in \mathsf{span}(\boldsymbol{\varphi}_1, \ldots, \boldsymbol{\varphi}_n)$, let

$$\boldsymbol{x} = X_1\boldsymbol{\varphi}_1 + \cdots + X_n\boldsymbol{\varphi}_n$$

  for some coefficients $X_1, \ldots, X_n$. Then

$$\frac{1}{c}\langle \boldsymbol{x}, \boldsymbol{x}\rangle = \frac{1}{c}\sum_{m=1}^{n} |x_m|^2 = \sum_{k=1}^{n} |X_k|^2 = \langle \boldsymbol{X}, \boldsymbol{X}\rangle,$$

  where $\boldsymbol{x} = [x_1 \ \cdots \ x_n]^\mathsf{T}$ and $\boldsymbol{X} = [X_1 \ \cdots \ X_n]^\mathsf{T}$.

- **Some Trigonometric Values**:

$$\cos\left(\frac{\pi}{6}\right) = \sin\left(\frac{\pi}{3}\right) = \sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2} \qquad \sin\left(\frac{\pi}{6}\right) = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2} \qquad \cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2}.$$

$$\sin\left(\frac{\pi}{2}\right) = 1 \qquad \cos\left(\frac{\pi}{2}\right) = \cos\left(\frac{3\pi}{2}\right) = 0 \qquad \sin\left(\frac{3\pi}{2}\right) = -1.$$

- **Some Perfect Squares**:

  $11^2 = 121, \quad 12^2 = 144, \quad 13^2 = 169, \quad 14^2 = 196, \quad 17^2 = 289, \quad 18^2 = 324, \quad 19^2 = 361.$

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**E2.1 (35 Points) Sum of Two Periodic Signals**

Consider a discrete-time signal $z : \mathbb{Z} \to \mathbb{R}$ such that

$$\forall n \in \mathbb{Z}, \qquad z[n] = x[n] + y[n].$$

The periodic signal $x : \mathbb{Z} \to \mathbb{R}$ has fundamental period $p$, and the signal $y : \mathbb{Z} \to \mathbb{R}$ has fundamental period $q$, where $p, q \in \{1, 2, 3, \ldots\}$. That is,

$$x[n+p] = x[n] \qquad \forall n \in \mathbb{Z}, \exists p \in \{1, 2, 3, \ldots\},$$

and

$$y[n+q] = y[n] \qquad \forall n \in \mathbb{Z}, \exists q \in \{1, 2, 3, \ldots\}.$$

(a) (10 Points) Suppose $p$ and $q$ are *coprime*—that is, they have no common positive divisor other than 1. Show that the signal $z$ is guaranteed to be periodic by determining a positive integer $r$, in terms of $p$ and $q$, such that $z[n+r] = z[n]$ for all integers $n$.

Let $r = pq$. Look at $z[n+r] = x[n+r] + y[n+r] \Longrightarrow$ $z[n+pq] = x[n+pq] + y[n+pq]$. But $x[n+pq] = x[n]$ since $x$ is $p$-periodic. Similarly, $y[n+pq] = y[n]$ since $y$ is $q$-periodic. Therefore, $z[n+pq] = z[n] \Longrightarrow z[n+r] = z[n]$ $\forall n \in \mathbb{Z}$

Absent additional info about $x$ & $y$, $pq$ is the smallest integer guaranteed to serve as a period for $z$.

(b) (10 Points) Suppose $p$ and $q$ are both *even* positive integers, such that $p = 2k$ and $q = 2\ell$, where $k$ and $\ell$ are coprime. Determine, in terms $k$, and $\ell$, the smallest positive integer $r$ such that the signal $z$ is guaranteed to be periodic—that is, $z[n+r] = z[n]$ for all integers $n$.

$p = 2k$ & $q = 2\ell \Longrightarrow$ Let $r = 2k\ell$ and look at $z[n+r]$.

$$\begin{aligned} z[n+r] = z[n+2k\ell] &= x[n+(2k)\ell] + y[n+(2\ell)k] \\ &= x[n+p\ell] + y[n+qk] \\ &= x[n] + y[n] = z[n] \quad \forall n \in \mathbb{Z} \end{aligned}$$

Since $k, \ell$ are coprime, $2k\ell$ is the smallest integer guaranteed to serve as a period for $z$, absent additional info about $x$ & $y$.

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**E2.1 (Continued)**

(c) (15 Points) Let $x[n] = \cos\left(\frac{3\pi}{4}n\right)$ and $y[n] = \sin\left(\frac{3\pi}{5}n\right)$, $\forall n \in \mathbb{Z}$. The fundamental period of $x$ is $p = 8$ samples.

(i) (5 Points) Determine $q$, the *fundamental* period of the signal $y$.

$$\sin\left(\frac{3\pi}{5}(n+q)\right) = \sin\left(\frac{3\pi}{5}n\right) \quad \forall n \in \mathbb{Z} \Rightarrow$$

$$\frac{3\cancel{\pi}}{5} q = 2\cancel{\pi} m, \exists m \in \mathbb{Z} \Rightarrow q = \frac{10}{3} m \Rightarrow \text{choose } m = 3,$$

(smallest possible integer)

so $\underline{q = 10}$

(ii) (10 Points) Show that the signal $z$ defined by

$$z[n] = x[n] + y[n] = \cos\left(\frac{3\pi}{4}n\right) + \sin\left(\frac{3\pi}{5}n\right), \qquad \forall n \in \mathbb{Z}$$

is periodic. As part of your work, you must determine the *fundamental* period $r$ of the signal $z$.

$p = 8 = 2k \Rightarrow k = 4$ & $q = 10 = 2\ell \Rightarrow \ell = 5$

Note that $k = 4$ an $\ell = 5$ are coprime, so pick

$\underbrace{r = 2k\ell = 2 \cdot 4 \cdot 5 = 40}_{\text{See also E2.1 (b)}} \Rightarrow \boxed{r = 40 \text{ samples Fundamental Period}}$

$$z[n+40] = \cos\left(\frac{3\pi}{4}(n+40)\right) + \sin\left(\frac{3\pi}{5}(n+40)\right)$$

$$= \cancel{\cos\left(\frac{3\pi}{4}n + 30\pi\right)} + \cancel{\sin\left(\frac{3\pi}{5}n + 24\pi\right)} = z[n]$$

$= \cos\left(\frac{3\pi}{4}n\right)$ $= \sin\left(\frac{3\pi}{5}n\right)$

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**E2.2 (30 Points) Signal Fusion in Self-Driving Cars**

A self-driving car receives the following types of sensor signals from its environment:

1. **LiDAR**, which stands for *Light Detection and Ranging*, is a remote-sensing method that uses pulsed laser to detect and measure distances to surrounding objects—in short, to create a 3D map of the environment.
2. **Radar**, which stands for *Radio Detection and Ranging*, uses radio waves to measure the positions, distances, velocities, and trajectories of surrounding objects.
3. **GPS**, which stands for *Global Positioning System*, uses satellite signals to geolocate the vehicle.

The following vectors in $\mathbb{R}^3$ represent these sensor signals, respectively:

$$\boldsymbol{a}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \boldsymbol{a}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \boldsymbol{a}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

The sensor signals contain overlapping information, which leads to interference and processing inefficiencies. To improve signal discrimination, your task is to curate these vectors using *Gram-Schmidt Orthogonalization*.

(a) (15 Points) **Measure of Data Overlap or Similarity: Angles Between Sensor Signals** For each pair of vectors $\boldsymbol{a}_i$ and $\boldsymbol{a}_j$, $(i \neq j)$, compute $\cos(\theta_{ij})$, where

$$\cos\theta_{ij} = \frac{\langle \boldsymbol{a}_i, \boldsymbol{a}_j \rangle}{\|\boldsymbol{a}_i\|\|\boldsymbol{a}_j\|}.$$

Which signal pairs have largest positive or negative overlap? Which the least?

$$\|\underline{a}_1\|^2 = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = 2 \Rightarrow \|\underline{a}_1\| = \sqrt{2}. \text{ Similarly, } \|\underline{a}_2\| = \|\underline{a}_3\| = \sqrt{2}$$

$$\langle \underline{a}_1, \underline{a}_2 \rangle = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = 1\cdot 1 + 1\cdot(-1) + 0\cdot 0 = 0 \Rightarrow \underline{a}_1 \perp \underline{a}_2$$

$$\langle \underline{a}_1, \underline{a}_3 \rangle = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = 1\cdot 1 + 1\cdot 0 + 0\cdot 1 = 1 \Rightarrow \cos\theta_{13} = \frac{1}{\sqrt{2}\sqrt{2}} = \frac{1}{2}$$

$$\langle \underline{a}_2, \underline{a}_3 \rangle = \begin{bmatrix} -1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = (-1)\cdot 1 + 1\cdot 0 + 0\cdot 1 = -1 \Rightarrow \cos\theta_{23} = \frac{-1}{\sqrt{2}\sqrt{2}} = \frac{-1}{2}$$

$a_1$ & $\underline{a}_2$ have zero overlap.
$a_1$ & $\underline{a}_3$ have [largest] positive overlap.
$\underline{a}_2$ & $\underline{a}_3$ have [largest] negative overlap.

The problem didn't ask for these, but here are the angles:

$\theta_{12} = \pi/2$ $\theta_{23} = 2\pi/3$
$\theta_{13} = \pi/3$

**E2.2 (Continued)**

(b) (15 Points) **Orthogonalization of the Sensor Signals**

Apply the Gram-Schmidt Orthogonalization process to obtain orthonormal vectors $\boldsymbol{q}_1$, $\boldsymbol{q}_2$, and $\boldsymbol{q}_3$.

$$\underline{z}_1 = \underline{a}_1 \Rightarrow \underline{q}_1 = \frac{\underline{z}_1}{\|\underline{z}_1\|} = \frac{\underline{a}_1}{\|\underline{a}_1\|} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\\0\end{bmatrix} \Rightarrow \boxed{\underline{q}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\\0\end{bmatrix} = \begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\\0\end{bmatrix}}$$

Since $\underline{a}_2$ is orthogonal to $\underline{a}_1$ already, we need only to normalize $\underline{a}_2$ to obtain $\underline{q}_2$:

$$\underline{q}_2 = \frac{\underline{a}_2}{\|\underline{a}_2\|} = \frac{1}{\sqrt{2}}\begin{bmatrix}-1\\1\\0\end{bmatrix} \longrightarrow \boxed{\underline{q}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix}-1\\1\\0\end{bmatrix} = \begin{bmatrix}-1/\sqrt{2}\\1/\sqrt{2}\\0\end{bmatrix}}$$

$$\underline{z}_3 = \underline{a}_3 - \langle \underline{a}_3, \underline{q}_1\rangle \underline{q}_1 - \langle \underline{a}_3, \underline{q}_2\rangle \underline{q}_2$$

$$\langle \underline{a}_3, \underline{q}_1\rangle = \begin{bmatrix}1 & 0 & 1\end{bmatrix}\begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\\0\end{bmatrix} = \frac{1}{\sqrt{2}}$$

$$\langle \underline{a}_3, \underline{q}_2\rangle = \begin{bmatrix}1 & 0 & 1\end{bmatrix}\begin{bmatrix}-1/\sqrt{2}\\1/\sqrt{2}\\0\end{bmatrix} = \frac{-1}{\sqrt{2}}$$

$$\underline{z}_3 = \begin{bmatrix}1\\0\\1\end{bmatrix} - \frac{1}{\sqrt{2}}\begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\\0\end{bmatrix} + \frac{1}{\sqrt{2}}\begin{bmatrix}-1/\sqrt{2}\\1/\sqrt{2}\\0\end{bmatrix} = \begin{bmatrix}1\\0\\1\end{bmatrix} - \begin{bmatrix}1/2\\1/2\\0\end{bmatrix} + \begin{bmatrix}-1/2\\1/2\\0\end{bmatrix} = \begin{bmatrix}0\\0\\1\end{bmatrix}$$

Since $\underline{z}_3$ is already normalized, our $\underline{q}_3$ is

$$\boxed{\underline{q}_3 = \begin{bmatrix}0\\0\\1\end{bmatrix}}$$

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**E2.3 (25 Points) Robot Motion in 2D**

A robot moves in the 2D $xy$-plane. Its state at any moment is represented as a vector in $\boldsymbol{s} \in \mathbb{R}^3$—namely, $\boldsymbol{s} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$, for some $x, y \in \mathbb{R}$, where the third coordinate is called a **homogeneous coordinate** to accommodate matrix transformations. Homogeneous coordinates are widely used in Computer Graphics, but you need not know *anything* about them to solve this problem. Simply treat the state vector $\boldsymbol{s}$ as a 3D vector, even though the robot's movements are restricted to the $xy$-plane.

The robot starts at position: $\boldsymbol{p} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$, and can undergo two types of transformations:

- **Rotation by Angle** $\theta = \frac{\pi}{2}$ **Counterclockwise** around the origin (as though you're looking down from the $z$-axis at the $xy$-plane):

$$\mathbf{R}\boldsymbol{s} = \begin{bmatrix} \cos\left(\frac{\pi}{2}\right) & -\sin\left(\frac{\pi}{2}\right) & 0 \\ \sin\left(\frac{\pi}{2}\right) & \cos\left(\frac{\pi}{2}\right) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} -y \\ x \\ 1 \end{bmatrix}.$$

- **Translation by** $(t_x, t_y) = (3, 2)$, meaning a shift of 3 units in the *positive* $x$-direction and 2 units in the *positive* $y$-direction:

$$\mathbf{T}\boldsymbol{s} = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x+3 \\ y+2 \\ 1 \end{bmatrix}.$$

**Important Note:** The translation is applied in the global coordinate frame—*not* relative to the robot's orientation. That is, no matter which way the robot faces, a translation of $(3, 2)$ always moves it 3 units in the positive $x$-direction and 2 units in the positive $y$-direction.

(a) (10 Points) Compute the robot's final position vector $\boldsymbol{y} \in \mathbb{R}^3$ (i.e, including the homogeneous coordinate) if it **first** rotates by $\theta = \frac{\pi}{2}$ counterclockwise and **then** translates by $(t_x, t_y) = (3, 2)$. You may continue your work at the top of next page.

Rotation first, followed by translation is given by $V = TR$, which produces the vector $\underline{y}_{TR} = V\underline{p} = TR\underline{p}$.

$$V = TR = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 3 \\ 1 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

We've taken the linear-combination-of-the-columns-of-T view. The color coding indicates which entries of R select which columns of T.

**E2.3** **(a) (Continued)**

$$\underline{y}_{TR} = V\underline{p} = \begin{bmatrix} 0 & -1 & 3 \\ 1 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = 2 \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 1 \cdot \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} + 1 \cdot \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0-1+3 \\ 2+0+2 \\ 0+0+1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}$$

$$\boxed{\underline{y}_{TR} = TR\underline{p} = \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}}$$

(b) (10 Points) Compute the robot's final position vector $\boldsymbol{z} \in \mathbb{R}^3$ (i.e, including the homogeneous coordinate) if it **first** translates by $(t_x, t_y) = (3, 2)$ and then rotates by $\theta = \dfrac{\pi}{2}$ counterclockwise.

Translation first, followed by rotation is given by $W = RT$, so $\underline{y}_{RT} = W\underline{p} = RT\underline{p}$.

$$W = \underbrace{\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{R} \overbrace{\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}}^{T} = \begin{bmatrix} (-1)\begin{bmatrix} 0 & 1 & 2 \end{bmatrix} \\ (1)\begin{bmatrix} 1 & 0 & 3 \end{bmatrix} \\ (1)\begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

We've taken the linear-combination-of-the-rows-of-T view. The color coding indicates which entries of R select which rows of T.

$$\underline{y}_{RT} = W\underline{p} = RT\underline{p} = \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = (2)\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + (1)\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} + (1)\begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 5 \\ 1 \end{bmatrix} \implies$$

$$\boxed{\underline{y}_{RT} = RT\underline{p} = \begin{bmatrix} -3 \\ 5 \\ 1 \end{bmatrix}}$$

(c) (5 Points) Explain whether—and, if so, why—the order of the transformations (rotation followed by translation, or vice versa) matters.

The order matters because matrix multiplication is not commutative, in general. In our case, $TR \neq RT$.

**E2.4 (35 Points) Orthonormal-Basis Decomposition)**

Consider the vectors

$$\boldsymbol{\psi}_1 = \frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix}, \qquad \boldsymbol{\psi}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix}-1\\0\\1\end{bmatrix}, \qquad \text{and} \qquad \boldsymbol{\psi}_3 = \frac{1}{\sqrt{6}}\begin{bmatrix}1\\-2\\1\end{bmatrix}.$$

(a) (15 Points) Show that the set $\{\boldsymbol{\psi}_1, \boldsymbol{\psi}_2, \boldsymbol{\psi}_3\}$ forms an orthonormal basis for $\mathbb{R}^3$.

$$\underline{\psi}_1^T \underline{\psi}_1 = \frac{1}{\sqrt{3}}\begin{bmatrix}1 & 1 & 1\end{bmatrix}\frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix} = \frac{1}{3}(1^2+1^2+1^2) = 1$$

$$\underline{\psi}_2^T \underline{\psi}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix}-1 & 0 & 1\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}-1\\0\\1\end{bmatrix} = \frac{1}{2}\left((-1)^2+0^2+1^2\right) = 1$$

$$\underline{\psi}_3^T \underline{\psi}_3 = \frac{1}{\sqrt{6}}\begin{bmatrix}1 & -2 & 1\end{bmatrix}\frac{1}{\sqrt{6}}\begin{bmatrix}1\\-2\\1\end{bmatrix} = \frac{1}{6}\left[1^2+(-2)^2+1^2\right] = 1$$

$$\underline{\psi}_1^T \underline{\psi}_2 = \frac{1}{\sqrt{6}}\begin{bmatrix}1 & 1 & 1\end{bmatrix}\begin{bmatrix}-1\\0\\1\end{bmatrix} = \frac{1}{\sqrt{6}}\left[1\cdot(-1)+1\cdot 0+1\cdot 1\right] = 0$$

$$\underline{\psi}_1^T \underline{\psi}_3 \propto \begin{bmatrix}1 & 1 & 1\end{bmatrix}\begin{bmatrix}1\\-2\\1\end{bmatrix} = 1\cdot 1+1\cdot(-2)+1\cdot 1 = 0$$

$$\underline{\psi}_2^T \underline{\psi}_3 \propto \begin{bmatrix}-1 & 0 & 1\end{bmatrix}\begin{bmatrix}1\\-2\\1\end{bmatrix} = (-1)\cdot 1+0\cdot(-2)+1\cdot 1 = 0$$

We've established that three vectors in $\mathbb{R}^3$ are mutually orthagonal $\Rightarrow$ They're linearly independent $\Rightarrow$ They form a basis in $\mathbb{R}^3$.

We've also shown that each vector has unit norm $\Rightarrow$ $\{\underline{\psi}_1, \underline{\psi}_2, \underline{\psi}_3\}$ is an orthonormal set.

So, we have an orthonormal basis in $\mathbb{R}^3$.

**E2.4 (Continued)**

(b) (15 Points) Consider the vector $\boldsymbol{x} = \begin{bmatrix} 3 \\ 4 \\ 12 \end{bmatrix}$.

Determine the coefficients $X_1$, $X_2$, and $X_3$ in the expansion $\boldsymbol{x} = X_1\boldsymbol{\psi}_1 + X_2\boldsymbol{\psi}_2 + X_3\boldsymbol{\psi}_3$.

$$\langle \underline{x}, \underline{\psi}_k \rangle = X_k \cancelto{1}{\langle \underline{\psi}_k, \underline{\psi}_k \rangle} \Longrightarrow X_k = \langle \underline{x}, \underline{\psi}_k \rangle \qquad k = 1, 2, 3$$

$$X_1 = \langle \underline{x}, \underline{\psi}_1 \rangle = \underline{x}^T \underline{\psi}_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} 3 & 4 & 12 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{3}}(3+4+12) = \frac{19}{\sqrt{3}}$$

$$X_2 = \langle \underline{x}, \underline{\psi}_2 \rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 3 & 4 & 12 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}}(-3+0+12) = \frac{9}{\sqrt{2}}$$

$$X_3 = \langle \underline{x}, \underline{\psi}_3 \rangle = \frac{1}{\sqrt{6}} \begin{bmatrix} 3 & 4 & 12 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{6}}(3-8+12) = \frac{7}{\sqrt{6}}$$

$$\boxed{\underline{x} = \frac{19}{\sqrt{3}} \underline{\psi}_1 + \frac{9}{\sqrt{2}} \underline{\psi}_2 + \frac{7}{\sqrt{6}} \underline{\psi}_3}$$

(c) (5 Points) Determine $\|\boldsymbol{X}\|$, the 2-norm of the coefficient vector $\boldsymbol{X} = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$.

**Note**: You may compute $\|X\|$ *even* if you're unsure of your values for $X_1$, $X_2$, and $X_3$.

Based on Parseval's Identity, where $\|\underline{\psi}_k\|^2 = c = 1$, we have $\langle \underline{x}, \underline{x} \rangle = \langle \underline{X}, \underline{X} \rangle \Rightarrow \|\underline{X}\|^2 = \|\underline{x}\|^2 = 3^2 + 4^2 + 12^2 = 169 \Rightarrow \boxed{\|\underline{X}\| = 13}$

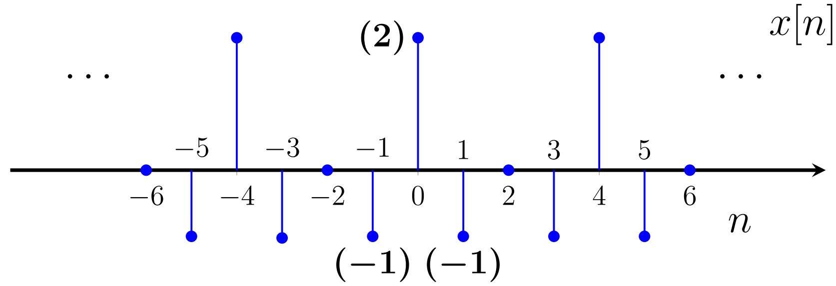
Note: We've used the Parseval's Identity $\frac{1}{c}\langle \underline{x}, \underline{x} \rangle = \langle \underline{X}, \underline{X} \rangle$ where $c = \|\underline{\psi}_k\|^2$, $k = 1, 2, 3$. Of course, we could've computed

$$\langle \underline{X}, \underline{X} \rangle = \underline{X}^T \underline{X} = \begin{bmatrix} 19/\sqrt{3} & 9/\sqrt{2} & 7/\sqrt{6} \end{bmatrix} \begin{bmatrix} 19/\sqrt{3} \\ 9/\sqrt{2} \\ 7/\sqrt{6} \end{bmatrix} = \frac{19^2}{3} + \frac{9^2}{2} + \frac{7^2}{6} = \frac{1014}{6} = 169$$

$\Rightarrow \|\underline{X}\| = 13$ Harder Way!

**E2.5 (35 Points) Discrete-Time Fourier Series (DTFS)**

Consider the discrete-time periodic signal shown below.




The signal has fundamental period $p = 4$ samples.

(a) (20 Points) Determine all the DTFS coefficients $X_k$, where $k \in \langle p \rangle = \langle 4 \rangle$, for the signal $x$. Here, $\langle p \rangle$ denotes a set of $p$ contiguous integers, such as $\{0, 1, \ldots, p-1\}$, or $\{-1, 0, 1, \ldots, p-2\}$, or $\{1, 2, \ldots, p\}$, etc.

$P=4 \Rightarrow \omega_0 = \frac{2\pi}{p} = \frac{2\pi}{4} = \frac{\pi}{2}$

We must determine four DTFS coefficients.

$X_0 = \frac{1}{p} \sum_{n \in \langle p \rangle} x[n]$ avg value of the signal over one period.

$= \frac{1}{4}(-1+2-1+0) = 0 \Rightarrow \boxed{X_0 = 0}$

$$X_k = \frac{1}{p} \sum_{n \in \langle p \rangle} x[n] e^{-ik\omega_0 n} = \frac{1}{4} \sum_{n=-1}^{2} x[n] e^{-ik\frac{\pi}{2}n}$$

$$= \frac{1}{4}\left(\underbrace{x[-1]}_{-1} e^{ik\frac{\pi}{2}} + \underbrace{x[0]}_{2} + \underbrace{x[1]}_{-1} e^{-ik\frac{\pi}{2}} + \cancelto{0}{x[2]} e^{-ik\pi}\right)$$

$$X_k = \frac{1}{4}\left(2 - 2\cos\left(\frac{k\pi}{2}\right)\right) \Rightarrow X_k = \frac{1-\cos(k\pi/2)}{2} \quad k=-1,0,1,2$$

$\boxed{X_{-1} = \frac{1}{2} = X_1}$ $\boxed{X_2 = \frac{1-(-1)}{2} = \frac{2}{2} = 1 \Rightarrow X_2 = 1}$ $\boxed{X_0 = \frac{1-1}{2} = 0}$ as we determined earlier.

**E2.5 (Continued)**

(b) (15 Points) Show that the signal $x$ can be expressed as $x[n] = \alpha^n + \cos(\beta\pi n)$ for all integers $n$, and for some parameters $\alpha$ and $\beta$.

Determine $\alpha$ and $\beta$ numerically. Your answers must be in the simplest form possible, but *not* expressed as decimals.

$$x[n] = \sum_{k \in \langle p \rangle} X_k e^{ik\omega_0 n} = \sum_{k=-1}^{2} X_k e^{ik\frac{\pi}{2}n}$$

$$= \underbrace{X_{-1}}_{1/2} e^{-i\frac{\pi}{2}n} + \cancel{X_0}^{\,0} + \underbrace{X_1}_{1/2} e^{i\frac{\pi}{2}n} + \underbrace{X_2}_{1} e^{i\pi n}$$

$$= \underbrace{e^{i\pi n}} + \frac{e^{i\frac{\pi}{2}n} + e^{-i\frac{\pi}{2}n}}{2} = \underbrace{(-1)^n} + \cos\left(\frac{\pi}{2}n\right) \implies$$

$$\boxed{x[n] = (-1)^n + \cos\left(\frac{\pi}{2}n\right) \qquad \alpha = -1 \quad \beta = \frac{1}{2}}$$

Not asked in the problem, but we can easily verify the signal values:

$x[-1] = \cancel{(-1)^{-1}}^{-1} + \cos\cancel{\left(-\frac{\pi}{2}\right)}^{0} = -1$

$x[2] = \cancel{(-1)^2}^{1} + \cancel{\cos\pi}^{-1} = 0$

$x[0] = (-1)^0 + \cos(0) = 1 + 1 = 2$

$x[1] = \cancel{(-1)^1}^{-1} + \cancel{\cos\left(\frac{\pi}{2}\right)}^{0} = -1 + 0 = -1$

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**E2.6 (25 Points) Matching Null Spaces**

Consider a matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$. Prove that $\mathcal{N}(\mathbf{A}) = \mathcal{N}(\mathbf{A}^\mathsf{T}\mathbf{A})$.

To prove this result, you must establish two things:

(a) (10 Points) Show that $\mathcal{N}(\mathbf{A}) \subseteq \mathcal{N}(\mathbf{A}^\mathsf{T}\mathbf{A})$. That is, for any vector $\boldsymbol{x} \in \mathbb{R}^n$, if $\boldsymbol{x} \in \mathcal{N}(\mathbf{A})$, then $\boldsymbol{x} \in \mathcal{N}(\mathbf{A}^\mathsf{T}\mathbf{A})$.

Let $\underline{x} \in \mathcal{N}(A) \Rightarrow A\underline{x} = \underline{0} \Rightarrow A^T A \underline{x} = A^T \underline{0} = \underline{0} \Rightarrow$

$$\underline{x} \in \mathcal{N}(A^T A)$$

(b) (15 Points) Show that $\mathcal{N}(\mathbf{A}^\mathsf{T}\mathbf{A}) \subseteq \mathcal{N}(\mathbf{A})$. That is, for any vector $\boldsymbol{x} \in \mathbb{R}^n$, if $\boldsymbol{x} \in \mathcal{N}(\mathbf{A}^\mathsf{T}\mathbf{A})$, then $\boldsymbol{x} \in \mathcal{N}(\mathbf{A})$. It may help you to recall that, based on a one of the properties of a norm, $\|\mathbf{A}\boldsymbol{x}\|^2 = 0$ if, and only if, $\mathbf{A}\boldsymbol{x} = \mathbf{0}$.

$\underline{x} \in \mathcal{N}(A^T A) \Rightarrow A^T A \underline{x} = \underline{0} \Rightarrow \underline{x}^T A^T A \underline{x} = \underline{x}^T \underline{0} = 0 \Rightarrow$

scalar

$(A\underline{x})^T (A\underline{x}) = 0 \Rightarrow \|A\underline{x}\|^2 = 0 \Rightarrow \|A\underline{x}\| = 0 \Rightarrow A\underline{x} = \underline{0}$

$\Rightarrow \underline{x} \in \mathcal{N}(A)$