EECS 16A
Fall 2020

Some of the Proofs We Have Covered So Far

1. Note 3 | 3.1.1
Prove the following two definitions of Linear Dependence are equivalent:

**Definition 3.1:** A set of vectors \( \{ \vec{v}_1, ..., \vec{v}_n \} \) is linearly dependent if there exist scalars \( \alpha_1, ..., \alpha_n \) such that \( \alpha_1 \vec{v}_1 + ... + \alpha_n \vec{v}_n = 0 \) and not all \( \alpha_i \)'s are equal to zero.

**Definition 3.2:** A set of vectors \( \{ \vec{v}_1, ..., \vec{v}_n \} \) is linearly dependent if there exist scalars \( \alpha_1, ..., \alpha_n \) and an index \( i \) such that \( \vec{v}_i = \sum_{j \neq i} \alpha_j \vec{v}_j \). In words, a set of vectors is linearly dependent if one of the vectors could be written as a linear combination of the rest of the vectors.

2. Note 3 | 3.1.3
Prove the following theorem:

**Theorem 3.1:** If the system of linear equations \( A\vec{x} = \vec{b} \) has an infinite number of solutions, then the columns of \( A \) are linearly dependent.

3. Note 4 | Example 4.1 (Example of Constructive Proof)
Prove that span \( \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\} = \mathbb{R}^2 \)

4. Note 4 | Example 4.2 (Example of Proof By Contradiction)
Prove the following theorem by contradiction:

**Theorem 4.1:** If the columns of \( A \) in the system of linear equations \( A\vec{x} = \vec{b} \) are linearly dependent, then the system does not have a unique solution.

5. Note 4 | Example 4.3
Let \( \{ \vec{v}_1, \vec{v}_2, ..., \vec{v}_n \} \) be a set of linearly dependent vectors in \( \mathbb{R}^n \). Take any matrix \( A \in \mathbb{R}^{m \times n} \). Prove that the set of vectors \( \{ A\vec{v}_1, A\vec{v}_2, ..., A\vec{v}_n \} \) is linearly dependent.

6. Note 4 | Example 4.4 (Example of Direct Proof)
Assume that vectors \( \vec{v}_1, \vec{v}_2 \) and \( \vec{v}_1 + \vec{v}_2 \) are all solutions to the system of linear equations \( A\vec{x} = \vec{b} \). Prove that \( \vec{b} \) must be the zero vector.

7. Discussion 3A | Q1
Given some set of vectors \( \{ \vec{v}_1, \vec{v}_2, ..., \vec{v}_n \} \), show the following:

(a) \( \text{span} \{ \vec{v}_1, \vec{v}_2, ..., \vec{v}_n \} = \text{span} \{ \alpha \vec{v}_1, \vec{v}_2, ..., \vec{v}_n \} \), where \( \alpha \) is a non-zero scalar. In other words, we can scale our spanning vectors and not change their span.

(b) \( \text{span} \{ \vec{v}_1, \vec{v}_2, ..., \vec{v}_n \} = \text{span} \{ \vec{v}_1 + \vec{v}_2, \vec{v}_2, ..., \vec{v}_n \} \). In other words, we can replace one vector with the sum of itself and another vector and not change their span.

8. Discussion 3A | Q2 Part 3
The distributivity property of matrix-vector multiplication holds for any vectors and matrices. Show for general \( A \in \mathbb{R}^{2 \times 2} \) and \( \vec{v}_1, \vec{v}_2 \in \mathbb{R}^2 \) that \( A(\vec{v}_1 + \vec{v}_2) = A\vec{v}_1 + A\vec{v}_2 \).
9. Note 6 | 6.1.1

Prove the following theorems:

(a) **Theorem 6.1**: If $A$ is an invertible matrix, then its inverse must be unique.

(b) **Theorem 6.2**: If $QP = I$ and $RQ = I$, then $P = R$. The matrix $P$ can be thought of as the “right” inverse of $Q$ and the matrix $R$ can be thought of as the “left” inverse of $Q$.

10. Note 6 | 6.2

Prove the following theorems:

(a) **Theorem 6.3**: If a matrix $A$ is invertible, there exists a unique solution to the equation $A\mathbf{x} = \mathbf{b}$ for all possible vectors $\mathbf{b}$.

(b) **Theorem 6.4**: If a matrix $A$ is invertible, its columns are linearly independent.

11. Homework 4 | Problem 6(f)

Consider a system consisting of $k$ reservoirs such that the entries of each column in the system’s state transition matrix sum to one.

Prove that if $s$ is the total amount of water in the system at timestep $n$, then total amount of water at timestep $n + 1$ will also be $s$.

12. Discussion 4B | Q3

Is the set $V = \left\{ \vec{v} \mid \vec{v} = c \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + d \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$, where $c, d \in \mathbb{R}$, a subspace of $\mathbb{R}^3$?

13. Note 9 | 9.6.1

Prove the following theorem:

**Theorem 9.1**: Given two eigenvectors $\vec{v}_1$ and $\vec{v}_2$ corresponding to two different eigenvalues $\lambda_1$ and $\lambda_2$ of a matrix $A$, it is always the case that $\vec{v}_1$ and $\vec{v}_2$ are linearly independent.

14. (Proof Out of Scope) Note 9 | 9.6.2 (Proof By Induction)

Prove the following theorem:

**Theorem 9.2**: Let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$ be eigenvectors of an $n \times n$ matrix with distinct eigenvalues. It is the case that all the $\vec{v}_i$ are linearly independent from one another.

The proof of this theorem is out of scope, but is presented anyway just for reference for those who are interested.